

Math 120ABC - Autumn 2002  
Mid-Term Exam Number Two  
November 21, 2002  
Solutions

1. Find the zeros and asymptotes, if any, of the following rational function:

$$f(x) = \frac{x^2 + 3x + 2}{2(x - 1)(x - 3)}.$$

Solution: We can factor the numerator:  $x^2 + 3x + 2 = (x + 2)(x + 1)$ . We see that the numerator and denominator have no factors in common. So, the zeros of the numerator (and therefore the zeros of the function) are  $x = -2$  and  $x = -1$ .

The zeros of the denominator determine the vertical asymptote(s). In this case, they are  $x = 1$  and  $x = 3$ .

To find the horizontal asymptote, we can write

$$\frac{x^2 + 3x + 2}{2(x - 1)(x - 3)} = \frac{x^2 + 3x + 2}{2(x^2 - 4x + 3)} = \frac{1 + \frac{3}{x} + \frac{2}{x^2}}{2(1 - \frac{4}{x} + \frac{3}{x^2})} \approx \frac{1}{2} \text{ for } |x| \text{ large.}$$

So  $y = \frac{1}{2}$  is the horizontal asymptote.

2. Suppose

$$f(x) = 3 + \frac{4x}{b + x}.$$

Find  $b$  so that

$$f^{-1}(x) = \frac{15 - 5x}{x - 7}.$$

Solution: Setting  $y = f(x)$  and solving for  $x$  we have:

$$y = 3 + \frac{4x}{b + x}$$

$$(y - 3)(b + x) = 4x$$

$$b(y - 3) + (y - 3)x - 4x = 0$$

$$b(y - 3) + x(y - 7) = 0$$

$$x = \frac{b(3 - y)}{y - 7}.$$

Thus,  $f^{-1}(x) = \frac{b(3 - x)}{x - 7}$ , so  $b$  must be 5 for this to be equal to  $f^{-1}(x) = \frac{15 - 5x}{x - 7}$ .

3. Give the multi-part rule for the following function:

$$g(x) = |x| + |x - 3|.$$

Solution: Since

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases},$$

we have

$$|x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 \leq 0 \end{cases} = \begin{cases} x - 3 & \text{if } x \geq 3 \\ -(x - 3) & \text{if } x \leq 3 \end{cases}.$$

So, when adding these functions together, there are three cases to consider:  $x < 0$ ,  $0 \leq x \leq 3$ ,  $x > 3$ .

If  $x < 0$ , then  $|x| = -x$  and  $|x - 3| = -(x - 3)$ , so  $|x| + |x - 3| = -2x + 3$ .

If  $0 \leq x \leq 3$ , then  $|x| = x$  and  $|x - 3| = -(x - 3)$ , so  $|x| + |x - 3| = 3$ .

If  $x > 3$ , then  $|x| = x$  and  $|x - 3| = x - 3$ , so  $|x| + |x - 3| = 2x - 3$ .

Putting this together, we have

$$|x| + |x - 3| = \begin{cases} -2x + 3 & \text{if } x \leq 0 \\ 3 & \text{if } 0 \leq x \leq 3 \\ 2x - 3 & \text{if } x \geq 3. \end{cases}$$

4. An antarctic explorer is stranded on a cylindrical iceberg. The top of the iceberg is circular and flat. Ocean currents in the area are causing the iceberg to stay in the same place, but rotate slowly. Because of this rotation, the explorer has a linear speed of 0.2 meters/second. A penguin standing at a distance of 30 meters from the center of the top surface of the iceberg has a linear speed of 0.17 meters/sec.

- (a) What is the angular velocity of the iceberg? Give the value in degrees per minute.
- (b) What is the farthest the penguin could be from the explorer?

Solution:

- (a) The information given about the penguin allows us to immediately find the angular speed of the penguin, which is the angular speed of the iceberg. Thus, using the formula  $v = r\omega$ ,

$$0.17 \text{ meters/sec} = (30 \text{ meters})\omega$$

so

$$\omega = \frac{0.17}{30} \text{ radians/sec} = 0.005666666 \text{ radians/sec.}$$

Converting to degrees per minute, we have the angular speed of the iceberg

$$\omega = 0.005666666 \frac{\text{radians}}{\text{sec}} \left( \frac{360 \text{ degrees}}{2\pi \text{ radians}} \right) \left( \frac{60 \text{ sec}}{1 \text{ minute}} \right) = 19.48056 \text{ degrees/minute.}$$

- (b) We know the distance of the penguin from the center of the top of the iceberg: 30 meters. To find the distance of the explorer from the center of the top of the iceberg, we use the formula  $v = r\omega$  again. We have

$$0.2 \text{ meters/second} = r(0.005666666 \text{ radians/sec})$$

so

$$r = 35.2941 \text{ meters.}$$

Thus the maximum distance between the penguin and the explorer is  $35.2941 + 30 = 65.2941$  meters.

5. Moose's head is a sphere that expands and contracts in such a way that the radius is a sinusoidal function of time. You start observing Moose's head at time  $t = 0$ . After 42 minutes, the radius reaches a maximum of 6.2 cm for the second time. It takes 20.2 minutes for the radius to expand from its smallest size, 6.0 cm, to its largest.

Find the function  $r(t)$  that describes the radius of Moose's head.

Solution: We are looking for  $A, B, C,$  and  $D$  so that

$$r(t) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

We have  $A = \frac{(\text{maximum } y\text{-value}) - (\text{minimum } y\text{-value})}{2} = \frac{6.2 - 6.0}{2} = 0.1.$

Also,  $D = \frac{(\text{maximum } y\text{-value}) + (\text{minimum } y\text{-value})}{2} = \frac{6.2 + 6.0}{2} = 6.1.$

Since it takes 20.2 minutes to go from a minimum to a maximum, and this is half the period  $B$ , we have  $B = (2)(20.2) = 40.4$ . Finally, we can determine  $C$  by subtracting  $B/4$  from the  $x$ -value of a maximum; here we get

$$C = 42 - \frac{40.4}{4} = 42 - 10.1 = 31.9.$$

Thus,

$$r(t) = 6.1 + 0.1 \sin\left(\frac{2\pi}{40.4}(t - 31.9)\right).$$

6. Find a value of  $x$  such that  $4 < x < 6$  and

$$4 + 3 \sin(2x - 1) = 5.$$

Solution: We can find a solution like this:

$$\sin(2x - 1) = \frac{1}{3}$$

$$2x - 1 = \sin^{-1} \frac{1}{3}$$

$$x = \frac{1 + \sin^{-1} \frac{1}{3}}{2} = 0.66991845.$$

Since

$$\sin(2x - 1) = \sin\left(\frac{2\pi}{\pi}\left(x - \frac{1}{2}\right)\right),$$

the next maximum of  $y = \sin(2x - 1)$  occurs at  $\pi/4 + 1/2 = 1.2853981633974\dots$ , so our "symmetric" solution is  $1.2853981633974 + (1.2853981633974 - 0.66991845) = 1.9008778$ . Since the period of  $\sin(2x - 1)$  is  $\pi$ , other solutions are  $3.8115111, 6.953103$  and  $5.0424704$  and we see that  $5.0424704$  satisfies  $4 < x < 6$  so this is the solution we were looking for.