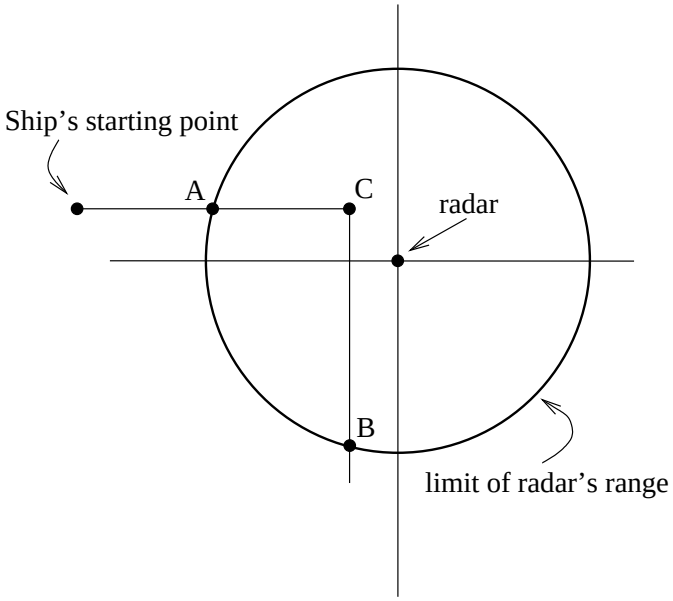
Math 120BDE - Autumn 2003
Mid-Term Exam Number One - Solutions
October 23, 2003

1. *A radar has a range of 45 km. A ship is traveling east at 25 km/h starting from a point 55 km west and 15 km north of the radar. After 2 hours, the ship changes course and begins traveling south at 25 km/h. For what length of time does the ship appear on the radar?*

If we set up the coordinates with the radar at the origin, and north in the vertical upward direction, we have a figure like the one at right. If we can find the coordinates of points A, B, and C, then we can find the appropriate distances and from that find the time spent on the radar.



The limit of the radar's range is represented by a circle with radius 45, centered at the origin. Its equation is thus

$$x^2 + y^2 = 45^2.$$

Since the ship begins travelling east, and its starting point is $(-55, 15)$, point A is the intersection of the line $y = 15$ with the circle. If we call A the point $(x, 15)$, then

$$x^2 + 15^2 = 45^2$$

i.e.,

$$x = \pm\sqrt{45^2 - 15^2} = \pm\sqrt{1800}.$$

Since the ship starts out west of the radar, we know we want the negative choice, i.e., A is the point $(-\sqrt{1800}, 15)$.

The ship travels east for 2 hours at 25 km/h to get to point C, so point C is

$$(-55 + (2)(25), 15) = (-5, 15).$$

Point B is then the intersection of the line $x = -5$ with the circle. The y-coordinate is found:

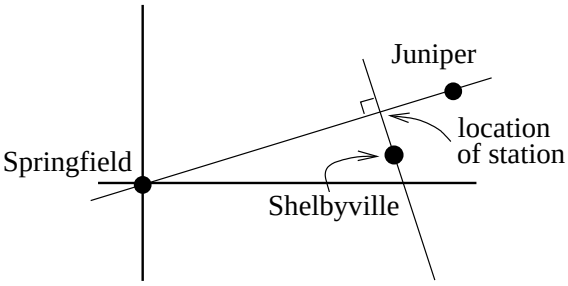
$$(-5)^2 + y^2 = 45^2$$

$$y = \pm\sqrt{45^2 - 5^2} = \pm\sqrt{2000}$$

Since the ship was sailing south, we know we want the negative choice: point B is $(-5, -\sqrt{2000})$.

So, the ship sailed $-5 - (-\sqrt{1800})$ km east on the radar, and then $15 - (-\sqrt{2000})$ km south on the radar. At 25 km/h, this means the ship was on the radar for $\dfrac{10 + \sqrt{1800} + \sqrt{2000}}{25}$, or about 3.8859106568 hours.

2. *A new high-speed railway line is being created to connect Springfield with Juniper. Juniper is 40 miles east and 35 miles north of Springfield, and the track for the railway will be laid in a straight line between the towns. Shelbyville is located 6 miles west and 15 miles south of Juniper. Where should a station on the railway be located to be as close as possible to Shelbyville? How far from Shelbyville would it be?*



We introduce a coordinate system with Springfield at the origin, and north being vertically upward. This gives the arrangement shown in the figure (which is not to scale). Juniper is located at the point $(40, 35)$, and Shelbyville is at the point $(34, 20)$. The track from Springfield to Juniper can be represented by the line:

$$y = \frac{35}{40}x = \frac{7}{8}x.$$

To find the location of the station, we need to find the intersection of the line which is perpendicular to the railway line and which passes through $(34, 20)$. This line will have slope $\frac{-8}{7}$, and so has equation

$$y - 20 = -\frac{8}{7}(x - 34)$$

i.e.,

$$y = -\frac{8}{7}(x - 34) + 20.$$

The intersection of these lines then can be found:

$$\frac{7}{8}x = -\frac{8}{7}(x - 34) + 20$$

$$x = \frac{\frac{8}{7} \cdot 34 + 20}{\frac{7}{8} + \frac{8}{7}} = \frac{3296}{113} \approx 29.16814159292.$$

So, the station should located at the point

$$(29.16814159292, 29.16814159292(\frac{7}{8})) \approx (29.1681416, 25.52212).$$

It will be

$$\sqrt{(34 - 29.1681416)^2 + (20 - 25.52212)^2} \approx 7.33762278$$

miles from Shelbyville.

3. *Let* $f(x) = x^2 - 3$, *and* $g(x) = |x| - \frac{1}{2}x$.

a. *Find* $g(f(x))$ *and write its multipart rule.*

Plugging one expression into the other, we have

$$g(f(x)) = |x^2 - 3| - \frac{1}{2}(x^2 - 3) = \begin{cases} x^2 - 3 - \frac{1}{2}(x^2 - 3) & \text{if } x^2 - 3 \geq 0, \\ -(x^2 - 3) - \frac{1}{2}(x^2 - 3) & \text{if } x^2 - 3 < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}x^2 - \frac{3}{2} & \text{if } x^2 \geq 3, \\ -\frac{3}{2}x^2 + \frac{9}{2} & \text{if } x^2 < 3 \end{cases}$$

$$= \begin{cases} \frac{1}{2}x^2 - \frac{3}{2} & \text{if } |x| \geq \sqrt{3}, \\ -\frac{3}{2}x^2 + \frac{9}{2} & \text{if } |x| < \sqrt{3} \end{cases}$$

$$= \begin{cases} \frac{1}{2}x^2 - \frac{3}{2} & \text{if } x \leq -\sqrt{3} \text{ or } x \geq \sqrt{3}, \\ -\frac{3}{2}x^2 + \frac{9}{2} & \text{if } -\sqrt{3} < x < \sqrt{3} \end{cases}$$

b. *Find all solutions to the equation*

$$g(f(x)) = 2.$$

We need to solve two equations:

$$\frac{1}{2}x^2 - \frac{3}{2} = 2 \text{ and } -\frac{3}{2}x^2 + \frac{9}{2} = 2.$$

The first gives solutions

$$\frac{1}{2}x^2 - \frac{3}{2} = 2$$
$$x^2 - 3 = 4$$
$$x^2 = 7$$
$$x = \pm\sqrt{7}.$$

The second gives solutions

$$-\frac{3}{2}x^2 + \frac{9}{2} = 2$$
$$-3x^2 + 9 = 4$$
$$-3x^2 = -5$$
$$x^2 = \frac{5}{3}$$
$$x = \pm\sqrt{\frac{5}{3}}.$$

4. *Suppose you run a movie theater and want to maximize the amount of money you take in with each showing of a movie. You have found that if you charge \$6 per ticket, you will sell 220 tickets, and if you charge \$9, you will sell 180 tickets. Assuming that the number of tickets you sell is a linear function of the ticket price, what price should you charge to get the most money?*

Since the number of tickets sold is a linear function of ticket price, we should find that function. Call it $T(x)$, where x is the ticket price. Then the graph of $y = T(x)$ is a line which passes through the points $(6, 220)$ and $(9, 180)$ so it has slope

$$\frac{220 - 180}{6 - 9} = -\frac{40}{3}$$

and so has equation

$$y - 220 = -\frac{40}{3}(x - 6)$$

i.e.,

$$T(x) = y = -\frac{40}{3}(x - 6) + 220 = -\frac{40}{3}x + 300.$$

Then, the amount of money made is $xT(x)$, i.e.,

$$xT(x) = -\frac{40}{3}x^2 + 300x.$$

This is a quadratic function. Completing the square, we have

$$xT(x) = -\frac{40}{3}\left(x^2 - \frac{45}{2}x\right) = -\frac{40}{3}\left(\left(x - \frac{45}{4}\right)^2 - \left(\frac{45}{4}\right)^2\right) = -\frac{40}{3}\left(x - \frac{45}{4}\right)^2 + \frac{3375}{2}.$$

The vertex is $(11.25, 1687.5)$, and since the coefficient of x^2 is negative, this yields the maximum value of the function. That is, the most money will be made if the ticket price is set at $\$11.25$.