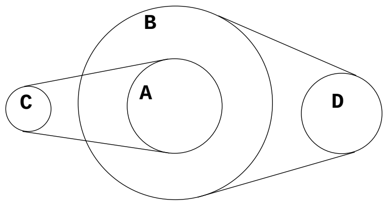
Math 120BDE - Autumn 2003
Mid-Term Exam Number Two Solutions
November 20, 2003

1. Four pulleys (A, B, C, and D) are attached by two belts as shown in the figure.



Pulleys A and B are rigidly attached to the same axle. The pulleys have radii as follows:

pulley	radius
A	0.85 cm
B	1.75 cm
C	0.4 cm
D	0.75 cm

Pulley A has an angular speed of 5 revolutions per minute.

(a) (5 points) What is the angular speed of pulley C?

Let v_C, r_C, and ω_C be the linear speed, radius and angular speed of pulley C, respectively. Similarym let v_A, r_A, and ω_A be the linear speed, radius and angular speed of pulley A, respectively. Then, since pulley A and pulley C are attached by a belt, their linear speeds are equal:

$$v_C = v_A.$$

The angular speed of pulley A is

$$\omega_A = 5\frac{\text{revolutions}}{\text{minute}}\left(\frac{2\pi\text{radians}}{\text{revolution}}\right) = 10\pi\frac{\text{radians}}{\text{minute}}$$

so the linear speed of pulley A is

$$v_A = r_A\omega_A = (0.85\text{ cm})10\pi\frac{\text{radians}}{\text{minute}} = 8.5\pi\frac{\text{cm}}{\text{minute}}.$$

So

$$v_C = 8.5\pi\frac{\text{cm}}{\text{minute}} = r_C\omega_C = (0.4\text{ cm})\omega_C$$

and hence

$$\omega_C = \frac{8.5\pi}{0.4}\frac{\text{radians}}{\text{minute}} = 21.25\pi\frac{\text{radians}}{\text{minute}}.$$

(b) (5 points) What is the linear speed of a point on the belt connecting pulley B and pulley D?

The linear speed of the belt is the linear speed of pulley B. Pulley B has the same angular speed as pulley A, 10π radians/minute. Its linear speed is

$$v_B = r_B\omega_B = (1.75\text{ cm})(10\pi\text{ radians/minute}) = 17.5\pi\frac{\text{cm}}{\text{minute}}.$$

2. A patient with sinusoidal fever has a temperature that is a sinusoidal function of time. At midnight, their temperature was dropping, and it kept dropping until it reached a minimum of $36°$C at 1:30 AM. It then climbed, and reached a maximum of $41°$C at 5 AM.

The patient needs surgery, but it is only safe to operate while the patient's temperature is below $40°$C. How long can the operation be?

The first step is to give a function for the patient's temperature. We find the mean to be $\frac{36+41}{2} = 38.5$, the amplitude to be $41 - 38.5 = 2.5$, the period to be $2(5-1.5) = 7$ hours, and the phase shift is $\frac{1.5+5}{2} = 3.25$.

So, t hours after midnight, the patient's temperature is

$$T(t) = 2.5 \sin\left(\frac{2\pi}{7}(t - 3.25)\right) + 38.5.$$

The next step is to find a time when the temperature is $40°$:

$$40 = 2.5 \sin\left(\frac{2\pi}{7}(t - 3.25)\right) + 38.5$$

$$\sin^{-1}(\frac{1.5}{2.5}) = \frac{2\pi}{7}(t - 3.25)$$

$$t = \frac{7}{2\pi}\sin^{-1}\frac{1.5}{2.5} + 3.25 = 3.96691\ldots .$$

This is the only time between 1:30 and 5 that the temperature is $40°$. By symmetry, the temperature was $40°$ at time

$$t = 1.5 - (3.96691\ldots - 1.5) = -0.96691\ldots .$$

At this time, the temperature was falling, and it fell until reaching the minimum of $36°$ at 1:30, then climbed, reaching $40°$ again at $t = 3.96691\ldots$. Thus, for each period, there is a span of

$$3.96691\ldots - (-0.96691\ldots) = 4.9338\ldots .$$

hours during which the operation could be performed.

3. Student X knows that their score on the final exam in Math 120 is a linear-to-linear function of the number of hours they study. That is, for h hours of study, student X will get a score of

$$s(h) = \frac{ah+b}{h+c}$$

for some constants a, b, and c. If they study zero hours, their score will be 35 percent. If they study 10 hours, their score will be 50 percent. The more student X studies, the closer their score will be to 100 percent, but they cannot get a score over 100 percent. How much should they study to get a score of 74 percent?

The function s has a horizontal asymptote of $y = a$, so from the description of the function, we must have $a = 100$.

Since $s(0) = 35$,

$$35 = \frac{b}{c}$$

so

$$b = 35c.$$

Since $s(10) = 50$,

$$50 = \frac{1000 + 35c}{10 + c}$$

from which we find $c = \frac{100}{3}$ and $b = \frac{3500}{3}$. Thus,

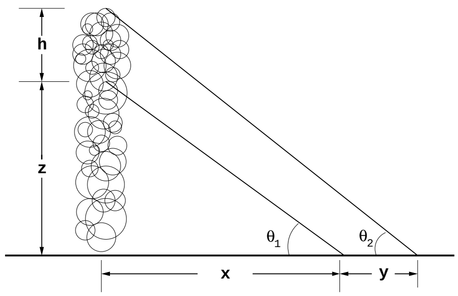
$$s(h) = \frac{100h + \frac{3500}{3}}{h + \frac{100}{3}}.$$

Setting this equal to 74 and solving for h yields

$$h = \frac{3900}{78} = 50\,.$$

So 50 hours of studying are needed to get a score of 74 percent on the final.

4. From your viewpoint, a vertically rising plume of smoke makes an angle of $54°$ with the horizontal. You decide to put more distance between yourself and the plume, and move 100 meters farther away. You measure the angle of view again and find it to be $48°$, but the plume has grown 20 meters taller in the time between your measurements. How tall is the plume?



Using the variables as in the diagram, we know $x = 100$ meters, $h = 20$ meters, $\theta_1 = 54°$, and $\theta_2 = 48°$. Using trigonometry, we have

$$\frac{z}{x} = \tan 54° \text{ and } \frac{h+z}{x+y} = \frac{20+z}{100+x} = \tan 48°$$

so

$$z = x \tan 54°$$

and

$$20 + x\tan 54° = (100 + x)\tan 48° = 100 \tan 48° + x \tan 48°$$

so

$$x = \frac{100\tan 48° - 20}{\tan 54° - \tan 48°}$$

and

$$z = \frac{100\tan 48° - 20}{\tan 54° - \tan 48°}\tan 54°.$$

So the height of the plume is

$$h + z = 20 + \frac{100\tan 48° - 20}{\tan 54° - \tan 48°}\tan 54° = 491.59326... \text{ meters.}$$