Exam Two Solutions

1 (a) One way to find the amplitude A is simply to take the maximum minus the mean. The mean here is $D = 700$ (we're told this), and the maximum is 1200. This means the amplitude is $A = 1200 - 700 = 500$ gallons.

(b) We've seen in the solution to part (a) that $A = 500$ and $D = 700$. The period is one year, or $B = 365$ days. Finally, we're told that the time at the maximum is $t_{\text{max}} = 213$ days, so one possible phase shift is $C = t_{\text{max}} - B/4 = 121.75$ days. Thus our formula is

$$s(t) = 500 \sin\left(\frac{2\pi}{365}(t - 121.75)\right) + 700.$$

(c) Now we are asked to find $s(137)$, the amount of ice cream on day 137 (May 17th). This is simply

$$\begin{aligned} s(t) &= 500 \sin\left(\frac{2\pi}{365}(137 - 121.75)\right) + 700 \\ &\approx 829.76 \text{ gallons.} \end{aligned}$$

(d) Now we wish to solve $s(t) = 800$, or

$$800 = 500 \sin\left(\frac{2\pi}{365}(t - 121.75)\right) + 700,$$

or

$$\frac{1}{5} = \sin\left(\frac{2\pi}{365}(t - 121.75)\right).$$

Thus one (principal) solution is

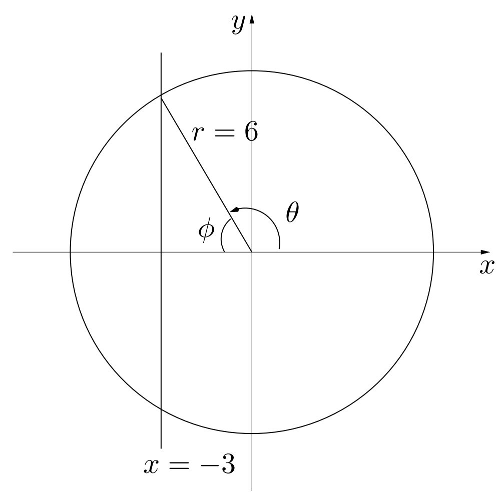
$$t = \frac{365}{2\pi} \sin^{-1}(1/5) + 121.75 \approx 133.45.$$

Another solution would be one period later: $t = 133.45 + 365 = 498.45$. Other solutions can be found using the symmetry solution:

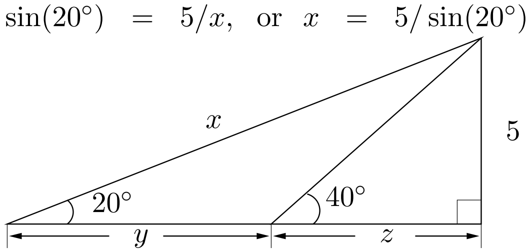
$$t = \frac{365}{2\pi}\left(\pi - \sin^{-1}(1/5)\right) + 121.75 \approx 292.55,$$

and from this we get yet another solution one period later: $t = 292.55 + 365 = 657.55$.

2 (a) I've added the angle ϕ in the circle to the right. From the picture, we see that $\theta + \phi = \pi$. Moreover, the triangle involving ϕ has side adjacent to ϕ has length 3 and hypotenuse has length 6. Thus $\cos(\phi) = 3/6$, or $\phi = \pi/3$. Hence $\theta = \pi - \pi/3 = 2\pi/3$.



(b) To find the length x, we look at the larger right triangle. From this triangle we see that $\sin(20°) = 5/x$, or $x = 5/\sin(20°) \approx 14.62$.



(c) To find the length y, we have added the notation "z" in the picture. We then get two relations involving tangent from the two right triangles:

$$\tan(20°) = \frac{5}{y+z} \qquad \tan(40°) = \frac{5}{z}.$$

Solving, we get

$$y + z = \frac{5}{\tan(20°)} \quad \text{and} \quad z = \frac{5}{\tan(40°)},$$

or, solving for y,

$$y = \frac{5}{\tan(20°)} - \frac{5}{\tan(40°)} \approx 7.78.$$

3 (a) We are asked for the linear speed v_B of wheel B. We find this using the formula $v = r\omega$ (which only applies if ω is measured in radians per time unit!). We're given $r_B = 10$ inches, and since wheels A and B are fastened at the axle, they have the same angular velocity. Thus

$$\omega_B = \omega_A = \left(150\ \frac{\text{revs}}{\text{min}}\right)\left(\frac{2\pi\ \text{rads}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right) = 5\pi\ \text{rads/sec}.$$

Thus the linear speed of wheel B is

$$v_B = (10\ \text{inches})\,(5\pi\ \text{rads/sec}) = 50\pi\ \text{in/sec} \approx 157.08\ \text{in/sec}.$$

(b) Now we're asked for wheel C's angular velocity. Using $v = r\omega$, we get $\omega = v/r$. Since wheels B and C are joined by a belt, they have the same linear speed: $v_C = v_B = 50\pi$ in/sec. Since r_C is given as 20 inches, we get

$$\omega_C = \frac{v_C}{r_C} = \frac{50\pi\ \text{in/sec}}{20\ \text{in}} = 5\pi/2\ \text{rads/sec}.$$

We convert this angular speed to RPM as follows:

$$\omega_C = \left(\frac{5\pi\ \text{rads}}{2\ \text{sec}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rads}}\right)\left(\frac{60\ \text{sec}}{1\ \text{min}}\right) = 75\ \text{RPM}.$$

This is the form of the answer we want.

4 (a) The zero is when $4 - 2x = 0$, or $(x, y) = (2, 0)$. The vertical asymptote is when $x + 5 = 0$, or $x = -5$. The horizontal asymptote can be found by multiplying the top and bottom by $1/x$ and letting x go off to infinity:

$$\frac{4-2x}{x+5} \cdot \frac{1/x}{1/x} = \frac{4/x - 2}{1 + 5/x} \longrightarrow \frac{0-2}{1+0} = -2.$$

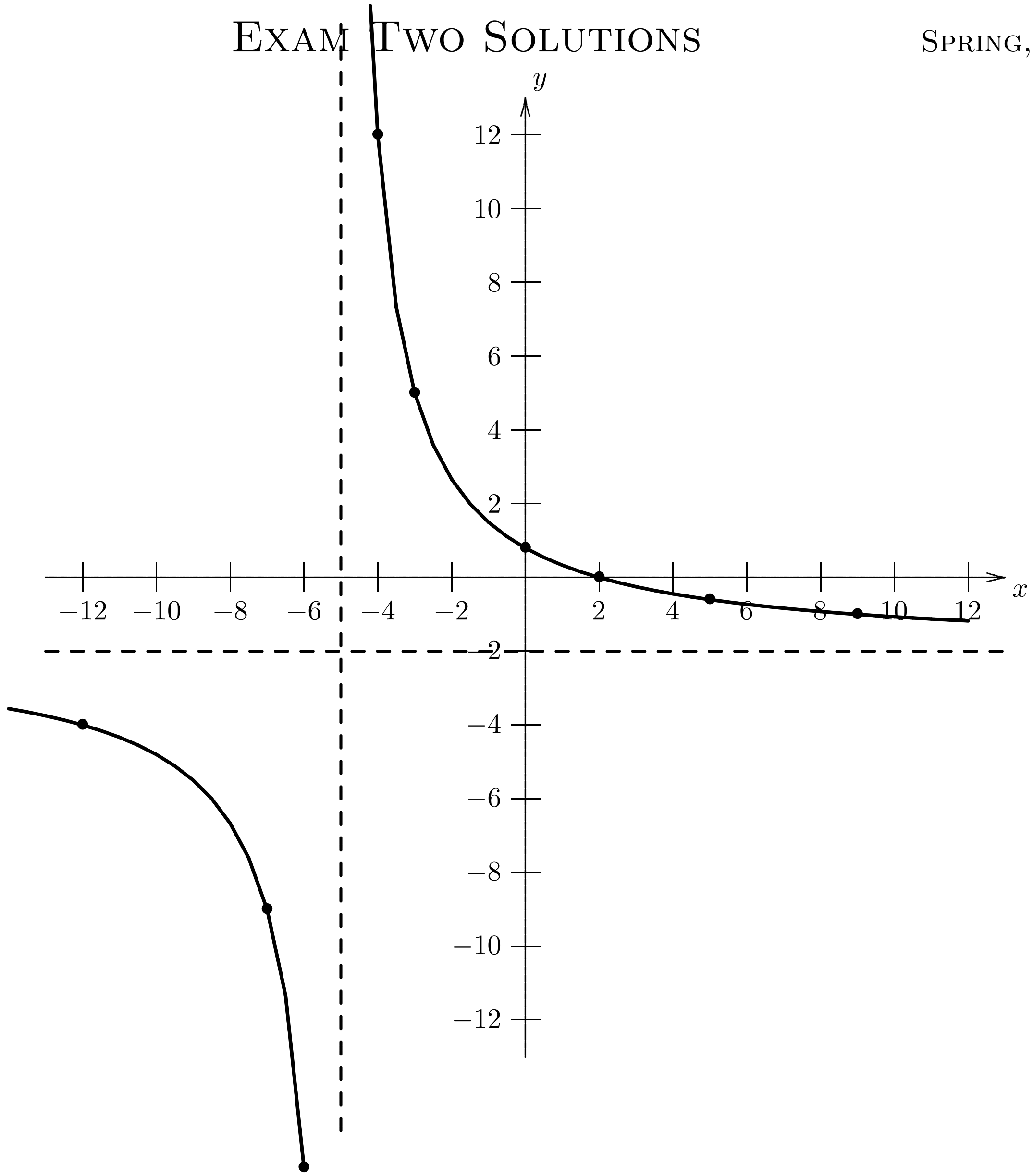
Thus $y = -2$ is the horizontal asymptote.

We plot the graph below, including many points (especially $(x, y) = (0, 4/5)$, where the curve crosses the y-axis). I've included dots at the following points:

$$(-12, -4) \quad (-7, -9) \quad (-6, -16) \quad (-4, 12) \quad (-3, 5)$$

$$(0, 4/5) \quad (2, 0) \quad (5, -3/5) \quad (9, -1)$$

You could, of course, plot many points other than these.



(b) To find $f^{-1}(x)$, we set $y = \frac{4-2x}{x+5}$ and solve for x:

$$\begin{aligned} y(x+5) &= 4-2x \\ xy+5y &= 4-2x \\ x(y+2) &= 4-5y \\ x &= \frac{4-5y}{y+2}. \end{aligned}$$

Thus $f^{-1}(x) = \dfrac{4-5x}{x+2}$. The domain of this function is all $x \neq -2$ and the range is all $y \neq -5$. (These are the range and domain, in that order, of the original function $y = f(x)$.)