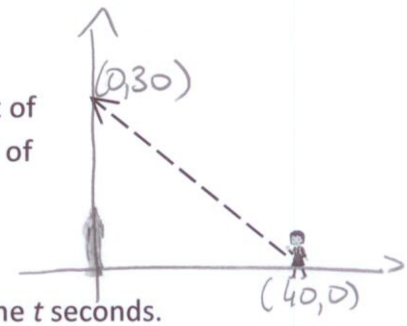


Problem 1 (16 pts) Linda is walking in a straight line from a point 40 feet due east of a statue, to a point 30 feet due north of the statue. She walks at a constant speed of 2 feet per second.

a) Impose a coordinate system with the origin at the statue and determine the parametric equations giving the x and y-coordinates for Linda's position at time t seconds.

From $(40,0)$ to $(0,30)$, Linda travels $d = \sqrt{(30)^2+(40)^2} = \boxed{50 \text{ feet}}$

At 2 ft/sec, it takes her $\Delta t = \frac{d}{v} = \frac{50 \text{ feet}}{2 \text{ ft/sec}} = \boxed{25 \text{ seconds}}$

Her coordinates at t seconds are:

$$\boxed{\begin{array}{l} x(t) = 40 - 1.6t \\ y(t) = 1.2t \end{array}}$$

since $x(0)=40$, and $V_x = \frac{\Delta x}{\Delta t} = \frac{0-40}{25} = \frac{-8}{5} = -1.6$

$y(0)=0$, $V_y = \frac{\Delta y}{\Delta t} = \frac{30}{25} = \frac{6}{5} = 1.2$

b) Find all the times when Linda is at a distance of 25 feet from the statue.

Distance Linda to statue at t seconds is: $d(t) = \sqrt{(x(t)-0)^2+(y(t)-0)^2}$

so $25 = \sqrt{(40-1.6t)^2+(1.2t)^2}$, which simplifies to

$25 = \sqrt{4t^2-128t+1600}$

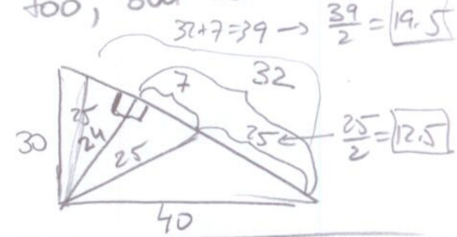
Squaring and simplifying:

$4t^2-128t+975=0$

Quadratic formula $t = \frac{128 \pm \sqrt{128^2-4(4)(975)}}{8}$

so $d(t)=25$ at $\boxed{t = 12.5 \text{ \& } 19.5 \text{ sec}}$

Note: this can be solved geometrically too, but harder.



c) Find Linda's position (x, y) when she is closest to the statue.

METHOD I: VERTEX OF PARABOLA

$d^2 = 4t^2 - 128t + 1600$

concave-up ⇒ min. at vertex

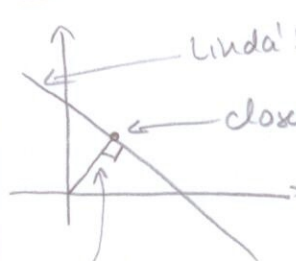
$t = -\frac{(-128)}{2(4)} = 16 \text{ sec}$

$x(16) = \dots = 14.4$

$y(16) = \dots = 19.2$

position $\boxed{(14.4, 19.2) \text{ feet}}$

METHOD II: PERPENDICULAR to PATH.



Linda's path: $y = -\frac{3}{4}x + 30$

closest to statue at intersection with line through origin perp. to path

$y = \frac{4}{3}x$ and $y = -3/4\, x + 30$

$\Rightarrow \frac{4}{3}x = -3/4\, x + 30 \Rightarrow x = 14.4$

$14.4 = x = 40 - 1.6t \Rightarrow t = 16$

$\Rightarrow y(16) = \dots = 19.2$

<u>Problem 2</u> (10 pts) Solve the following two equations. Show all steps, and box your final answers.

a) $7x + |2x - 5| = 4$

$$7x + |2x-5| = \begin{cases} 7x + (2x-5) & \text{if } 2x-5 \geq 0,\ 2x \geq 5,\ x \geq 5/2 \\ 7x - (2x-5) & \text{if } x \leq 5/2 \end{cases} = \begin{cases} 9x-5 & \text{if } x \geq 5/2 \\ 5x+5 & \text{if } x \leq 5/2 \end{cases}$$

<u>Case 1</u>: If $x \geq 5/2$ the equation is $9x - 5 = 4$

$9x = 9$

$x = 1$ ← <u>not</u> a sol. because it's not $\geq 5/2$

<u>Case 2</u>: If $x \leq 5/2$ then we solve: $5x + 5 = 4$

$5x = -1$

$x = -1/5$ ← it is $\leq 5/2$ so it is a sol.

$$\boxed{\text{Sol: } x = -1/5}$$

b) $ln\left(\frac{2x}{x-5}\right) = 3$

$$e^{\ln\left(\frac{2x}{x-5}\right)} = e^3$$

$$\frac{2x}{x-5} = e^3$$

$$2x = e^3(x-5)$$

$$2x - e^3 x = -5e^3$$

$$\boxed{x = \frac{-5e^3}{2-e^3} = \frac{5e^3}{e^3-2} \cong 5.55}$$

Problem 3 (12 points)

The population of Arcadia increases by 8% every 10 years. The population of Brom triples every 120 years.

The two cities had equal populations of 10,000 residents each in the year 2000.

In what year will the city of Brom have twice as many residents as the city of Arcadia?

$t =$ years after year 2000

Arcadia: $A(t) = A(0)\, a^t = 10{,}000\, a^t$

$t = 10$ years: $A(10) = 10{,}000 + 0.08(10{,}000)$
$= 10{,}800$

$10{,}800 = A(10) = 10{,}000\, a^{10}$

$1.08 = a^{10} \Rightarrow a = \sqrt[10]{1.08}$

so $\boxed{A(t) = 10{,}000\left(\sqrt[10]{1.08}\right)^t}$

Brom: $B(0) = 10{,}000 \Rightarrow B(t) = 10{,}000\, b^t$

After 120 years: $B(120) = 30{,}000 = 10{,}000\, b^{120}$

$3 = b^{120}$

$b = \sqrt[120]{3}$

so $\boxed{B(t) = 10{,}000\left(\sqrt[120]{3}\right)^t}$

Brom has twice the pop. of Arcadia when: $\boxed{B(t) = 2A(t)}$

i.e.: $10{,}000\left(\sqrt[120]{3}\right)^t = 20{,}000\left(\sqrt[10]{1.08}\right)^t$

$\left(\sqrt[120]{3}\right)^t = 2\left(\sqrt[10]{1.08}\right)^t$

$\left(\dfrac{\sqrt[120]{3}}{\sqrt[10]{1.08}}\right)^t = 2$

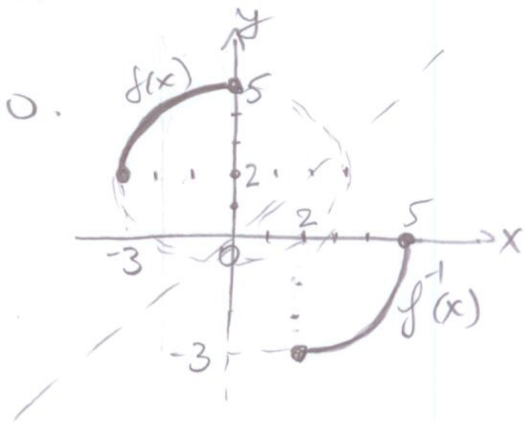
$t \ln\left(\dfrac{\sqrt[120]{3}}{\sqrt[10]{1.08}}\right) = \ln 2$

$t = \dfrac{\ln 2}{\ln\left(\dfrac{\sqrt[120]{3}}{\sqrt[10]{1.08}}\right)} \cong 475.08 \Rightarrow \boxed{\text{year } 2475}$

Problem 4 (12 pts)

a) (5 pts) Consider the function $f(x) = 2 + \sqrt{9 - x^2}$, restricted to domain $-3 \le x \le 0$. Find the rule for the inverse function, $f^{-1}(x)$.

$$y = 2 + \sqrt{9-x^2}, \quad -3 \le x \le 0.$$



$$y - 2 = \sqrt{9-x^2}$$

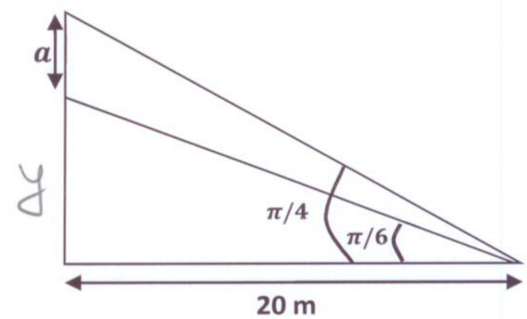
$$(y-2)^2 = 9 - x^2$$

$$x^2 = 9 - (y-2)^2$$

$$\left.\begin{array}{l} x = \pm\sqrt{9-(y-2)^2} \\ -3 \le x \le 0 \end{array}\right\} \Rightarrow x = -\sqrt{9-(y-2)^2}$$

$$\boxed{\begin{aligned} f^{-1}(x) &= -\sqrt{9-(x-2)^2} \\ &= -\sqrt{-x^2+4x+5} \end{aligned}}$$

b) (7 pts) Compute the length labeled "a" on the following picture (all given angles are in radians, so make sure your calculator is in radian mode.)



$$\tan\frac{\pi}{6} = \frac{y}{20} \Rightarrow y = 20\tan\frac{\pi}{6}$$

$$= 20\left(\frac{\sqrt{3}}{3}\right) \approx 11.547$$

$$\tan\frac{\pi}{4} = \frac{a+y}{20} \Rightarrow a + y = 20\tan(\pi/4)$$

$$a = 20\underbrace{\tan(\pi/4)} - y$$

$$= 20(1) - \frac{20\sqrt{3}}{3} = \boxed{\frac{20(3-\sqrt{3})}{3}} \text{ meters}$$

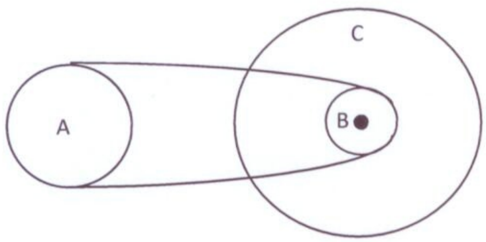
$$\approx 20 - 11.547 \approx \boxed{8.453} \text{ meters.}$$

Problem 5 (8 pts)

You are designing a system of wheels and belts as pictured. Wheels B and C are rigidly fastened to the same axle, while wheels A and B are connected by a belt.

You want wheel A to rotate at 2 rad/sec, and wheel C to have a linear velocity of 6 miles per hour. Wheel A has radius $R_A = 6$ in, and wheel C has radius $R_C = 15$ in.

What should be the radius of wheel B?



Know: $\omega_A = 2$ rad/sec
$R_A = 6$ in
$R_C = 15$ in
$v_C = 6$ mph $\Rightarrow v_C = 105.6$ in/sec

Want: $R_B = ?$

$\boxed{v_A = v_B \text{ (belt)}}$

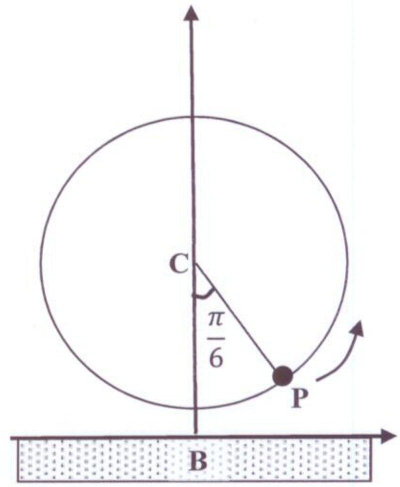
	Wheel A	Wheel B	Wheel C
$R\omega = V$	12 in	12 in	105.6 in/sec
$\frac{V}{\omega} = R$	6 in	?? $= \frac{12}{7.04} = 1.7$	15 in
$\frac{V}{R} = \omega$	2 rad/sec	7.04	7.04 rad/sec

$\boxed{\omega_B = \omega_C}$ (same axle)

$$\boxed{R_B \cong 1.7 \text{ in}}$$

Problem 6. (16 pts) Percy is riding on a ferris wheel of radius 50 feet, whose center C is 52 feet above ground. The wheel rotates at a constant rate, taking 1.5 minutes for each full revolution. The wheel starts turning when Percy is at the point P, making an angle of $\frac{\pi}{6}$ radians with the vertical, as shown.

(Make sure your calculator is in radian mode)



a) (5 pts) How high is Percy above ground when the wheel starts turning?

$y = 52 + 50\sin(-\pi/3)$ (or, from scratch $52 - 50\cos\left(\frac{\pi}{6}\right)$)

$= 52 - 50\frac{\sqrt{3}}{2}$

$\cong \boxed{8.7 \text{ feet}}$

b) (4 pts) Impose a coordinate system **with the origin at the base point B.** What is the equation of the line CP?

slope $= \tan(-\pi/3) = -\sqrt{3}$

y-intercept $= 52$

$\boxed{y = -\sqrt{3}x + 52}$

OR P has coordinates

$x_P = 50\cos(-\pi/3) = 25$

$y_P = 52 + 50\sin(-\pi/3) \cong 8.7$

$\Rightarrow$ slope CP $= \frac{y_P - y_C}{x_P - x_C} = \frac{8.7 - 52}{25 - 0} \cong -1.732$

$\boxed{y \cong -1.732x + 52}$

c) (7 pts) Percy drops his ice cream cone 1.25 minutes after the wheel starts moving. If the cone falls straight down from Percy's position at that time, where does it land with respect to the base point B?

$\omega = \frac{1 \text{ rev}}{1.5 \text{ min}} = \frac{2\pi \text{ rad}}{1.5 \text{ min}} = \frac{4\pi}{3} \text{ rad/min}$

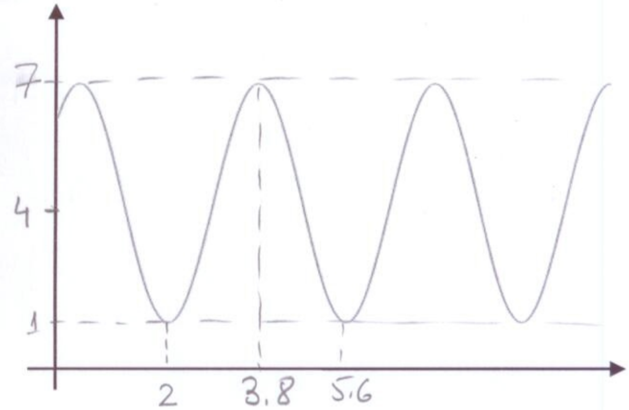
$x(t) = x_{center} + R\cos(\theta_0 + \omega t)$

$= \underbrace{0} + 50\cos\left(-\pi/3 + \frac{4\pi}{3} \times (1.25)\right)$

$= 50\cos\left(-\pi/3 + \frac{5\pi}{3}\right) = 50\cos\left(\frac{4\pi}{3}\right) = -25 \text{ feet}$

It lands $\boxed{\text{25 feet to the left of B.}}$

Problem 7 (16 pts) The **depth** of a swimming salmon below the water surface can be modeled by a sinusoidal function of time. The salmon's depth varies between a minimum of 1 foot and a maximum of 7 feet below the surface of the water. It takes the salmon 1.8 minutes to move from its minimum depth to its successive maximum depth, and it first reaches the minimum depth at $t = 2$ minutes.



a) Find the sinusoidal function $d(t) = A\sin\left(\frac{2\pi}{B}(t - C)\right) + D$ which models the depth of the salmon after t minutes.

$A = \frac{7-1}{2} = 3$, $D = \frac{7+1}{2} = 4$

$B = 2(1.8) = 3.6$, $C = 2 + \frac{3.6}{4} = 2.9$

$$\boxed{d(t) = 3\sin\left(\frac{2\pi}{3.6}(t-2.9)\right) + 4}$$

b) Compute all the times during the first 5 minutes when the depth of the fish is exactly 3 feet below the surface.

$$3\sin\left(\frac{2\pi}{3.6}(t-2.9)\right) + 4 = 3$$

$$\sin\left(\frac{2\pi}{3.6}(t-2.9)\right) = -1/3$$

PS: $\frac{2\pi}{3.6}(t-2.9) = \arcsin(-1/3)$

$t - 2.9 = \frac{3.6}{2\pi}\arcsin(-1/3)$

$t = 2.9 + \frac{3.6}{2\pi}\arcsin(-1/3)$

$\boxed{t \cong 2.705}$

If we add/subtract multiples of $B = 3.6$ we get times outside of first 5 minutes

S.S: $\frac{2\pi}{3.6}(t-2.9) = \pi - \arcsin(-1/3)$

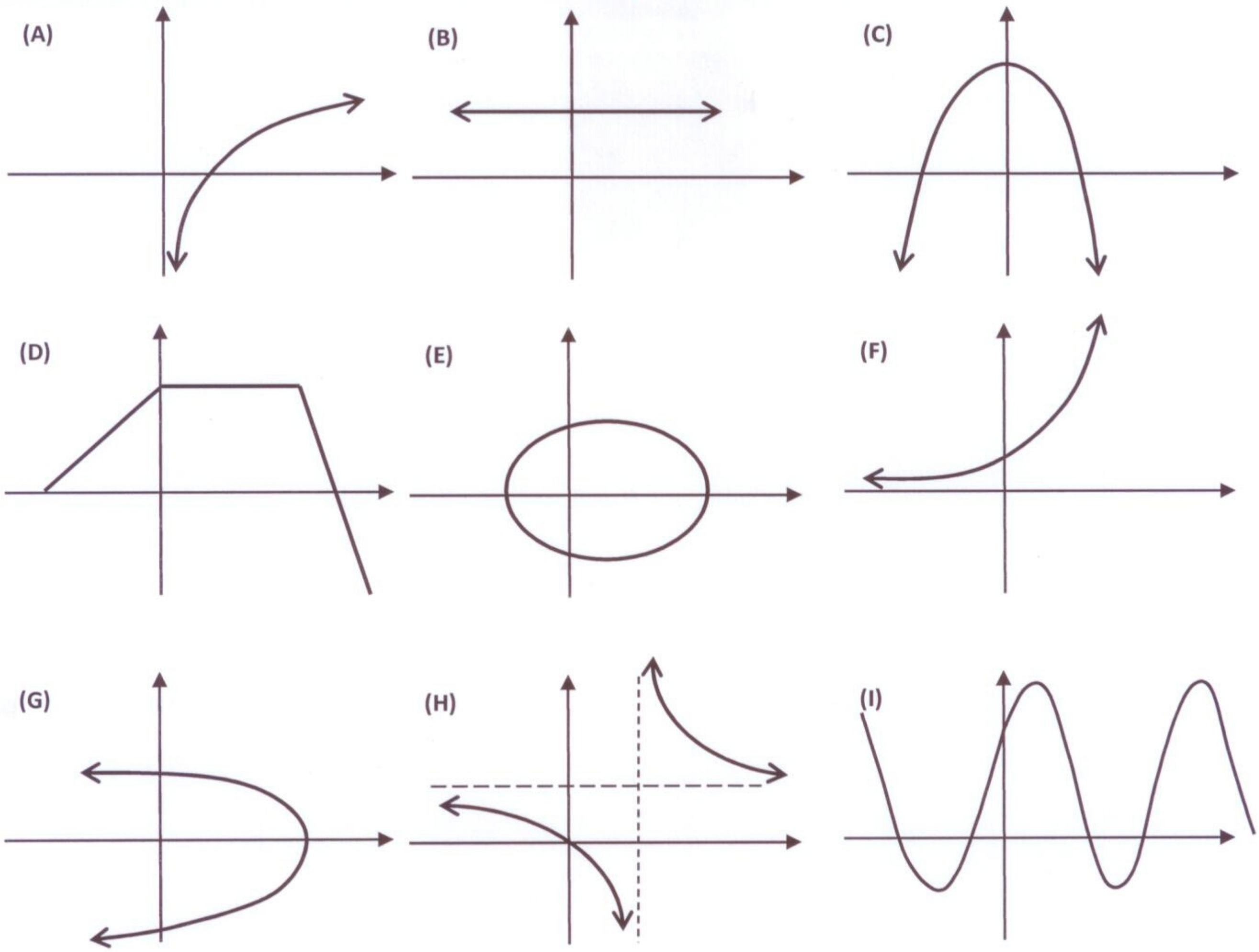
$t = 2.9 + \frac{3.6}{2\pi}(\pi - \arcsin(-1/3))$

$\boxed{t \cong 4.895}$

If we subtract $B = 3.6$ we get:

$\boxed{t \cong 1.295}$

Problem 8 (10 points) Consider the following nine graphs, labeled (A)-(I).



Answer the following questions -- no need to justify or show any work.

a) Which of these graphs above do **not** represent graphs of functions? E, G

b) From those that are functions, which ones **are one-to-one** (invertible) ? A, F, H

c) Which of the graphs shown above (if any) could be the graph of a function $y = f(x)$ of the following types:

i. Quadratic? C

ii. Exponential? F

iii. Logarithmic? A

iv. Rational (linear-to-linear)? H

v. Multi-part? D

vi. Trigonometric? I