Math 120B (Collingwood) SOLUTIONS
8 February 2000
Midterm #1 (100 points)

Instructions:

- Your exam contains 5 problems. The entire exam is worth 100 points. The point value of each problem is clearly marked.
- Your exam should contain 7 pages; please make sure you have a complete exam.
- Box in your final answer when appropriate. Use the back of your exam pages if you need extra room.
- When appropriate, carry out calculations to at least two decimal places.
- You have 50 minutes for this midterm. You **MUST** show work for credit. No credit for answers only (unless stated otherwise). You may use a graphing calculator to check yourself, but "zooming" is not sufficient justification for any answer on the exam. If in doubt, ask for clarification.
- Make sure to do your own work on the exam.
- Please sign the exam. In doing so, you understand that we may make photocopies of some exams prior to returning.

Signature_____

Problem #1 (36 pts) _____

Problem #2 (16 pts) _____

Problem #3 (12 pts) _____

Problem #4 (12 pts) _____

Problem #5 (24 pts) _____

TOTAL (100 pts) _____

Problem 1 (36pts). Computations. Show enough work to indicate how you got your answer. No credit for answer only. Box the answer you want graded. For this problem,

$$f(x) = 2x^2 - x - 1$$

$$g(x) = \frac{x}{2} + 3$$

(a) (8pts) Calculate and simplify as far as possible; there should NOT be an h in the denominator of the final answer.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 - (x+h) - 1 - (2x^2 - x - 1)}{h} \\
&= \frac{2(x^2 + 2xh + h^2) - x - h - 1 - 2x^2 + x + 1}{h} \\
&= \frac{2x^2 + 4xh + 2h^2 - x - h - 1 - 2x^2 + x + 1}{h} \\
&= \frac{4xh + 2h^2 - h}{h} \\
&= 4x + 2h - 1
\end{aligned}$$

(b) (2pts) Plug $h = 0$ into the your final expression for (a); what is the result?

$4x - 1$

(c) (8pts) Calculate and simplify as far as possible

$$g(f(x)) = g(2x^2 - x - 1) = (1/2)(2x^2 - x - 1) + 3 = x^2 - x/2 + 5/2$$

$$g(g(x)) = g((x/2) + 3) = (1/2)((x/2) + 3) + 3 = (x/4) + 9/2$$

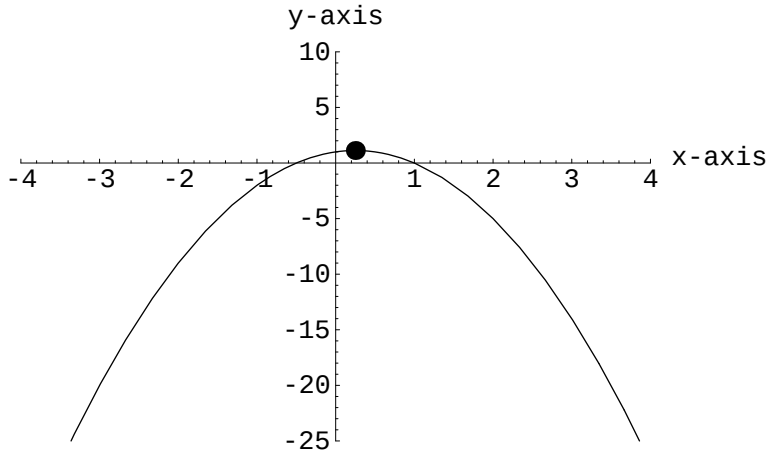
(Problem 1 Continued)

(d) (10pts) Put the function $y = -f(x)$ into vertex form and sketch its graph; make sure to label the vertex.

$y = -2x^2 + x + 1$, so $a = -2, b = 1$, which means $h = -b/2a = 1/4, k = -2(1/4)^2 + (1/4) + 1 = 9/8$. This means vertex form is

$$y = -2(x - \frac{1}{4})^2 + \frac{9}{8}.$$

So, vertex=$(1/4, 9/8)$.



(e) (8pts) Find the largest possible domain of the function $y = \pi\sqrt{-f(x)}$

Must have term under radical non-negative. From graph in (a), we need to see where graph crosses x-axis; these are the roots of the quadratic $y = -2x^2 + x + 1$. Can find these via quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-2)(1)}}{2(-2)} = -0.5, 1.$$

So, largest domain is

$$-1/2 \leq x \leq 1$$

Problem 2.(16pts) For this problem, consider the line with equation $y = -\frac{3}{4}x + 10$ and the circle of radius 5 centered at the point $(5, 0)$. Simultaneously solve the equation of the line and circle. How many times does the line intersect the circle? Draw a picture of the circle and the line in the coordinate system provided. Label any intersection point(s).

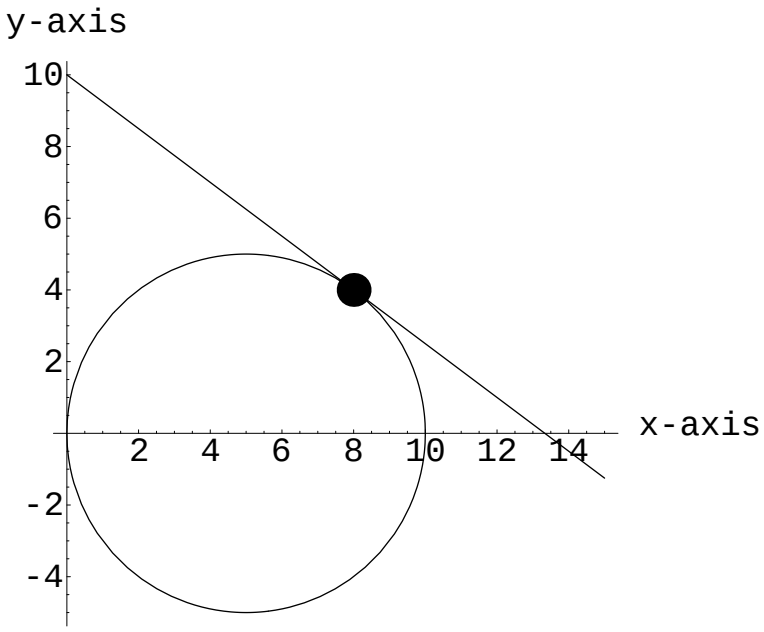
Simultaneously solve equations:

$(x-5)^2 + y^2 = 25$ and $y = -0.75x + 10$

to get

$$\begin{aligned}
(x-5)^2 + (-0.75x+10)^2 &= 25 \\
(25/16)x^2 - 25x + 125 &= 25 \\
x^2 - 16x + 64 &= 0 \\
(x-8)^2 &= 0.
\end{aligned}$$

So the only solution is $x = 8$, meaning the only intersection point is $(8, 4)$; i.e. theline crosses the circle ONCE.



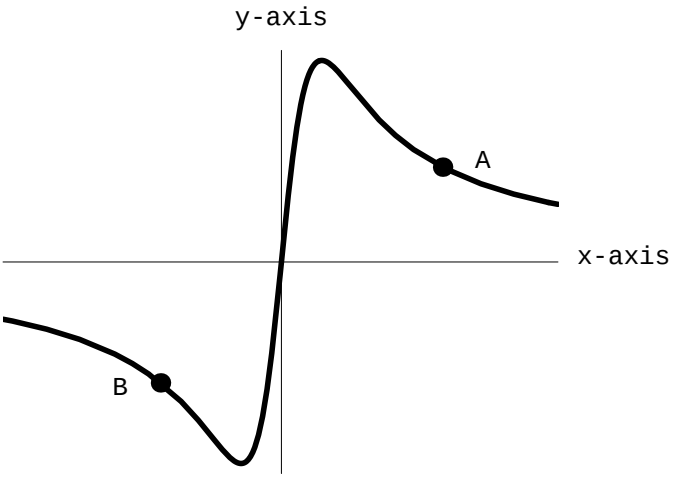
Problem 3.(12pts) For this problem,

$$y = f(x) = \frac{10x}{x^2+1}$$

whose graph is pictured on the domain

$$-6 \leq x \leq 6$$

. You must show your work; no credit for answers only. Box the answer you want graded.



(a) (3pts) The point labeled "A" has x-coordinate equal to 4; what is the y-coordinate of "A"?

$A = (4, f(4)) = (4, 40/17) = (4, 2.35)$.

(b) (6pts) The point labeled "B" has y-coordinate equal to -3; what is the x-coordinate of "B"?

Solve

$$\begin{aligned} -3 &= f(x) \\ -3 &= \frac{10x}{x^2+1} \\ -3x^2 - 3 &= 10x \\ 3x^2 + 10x + 3 &= 0 \\ x &= -3, -1/3 \end{aligned}$$

So, $B = (-3, -3)$ or $B = (-1/3, -3)$; must be $B = (-3, -3)$ from picture.

(c) (3pts) The minimum value of $f(x)$ occurs when $x = -1$ and the maximum value of $f(x)$ occurs when $x = 1$. What is the range of the function $y = f(x)$?

Range is

$$f(-1) \leq y \leq f(1)$$

which gives

$$-5 \leq y \leq 5$$

Problem 4.(12pts) Ted is selling tickets to a concert. From his past experience, he can sell 30 tickets if he charges \$6 a ticket. If he charges \$5 a ticket he sells 40 tickets. His expenses to print up posters amount to \$40.

1. (5pts) Give a linear function $n = f(t)$ relating the number of tickets sold n to the price of a ticket t. How much money will Ted take in if he prices tickets at \$8? Using this linear model, what ticket price will result in no ticket sales?

   In the tn-coord system, where input variable is t and output variable is n, we have two points on the linear graph of this linear function: $(5, 40)$ and $(6, 30)$. By the two point formula

   $$n = f(t) = \frac{40 - 30}{5 - 6}(t - 5) + 40 = -10(t - 5) + 40 = -10t + 90$$

   If he prices tickets at \$8, then he takes in

   $$\$8 f(8) = 8(-80 + 90) = \$80.$$

   Solving $f(t) = 0$, we get $0 = -10t + 90$, so \$9 a ticket yields no ticket sales.

2. (4pts) Give a function $p = g(t)$ relating Ted's profit p to the price of a ticket t. (Remember to subtract his expenses.)

   profit = (total amount taken in) - (expenses)
   $p = g(t) = tf(t) - 40 = t(-10t + 90) - 40 = -10t^2 + 90t - 40$

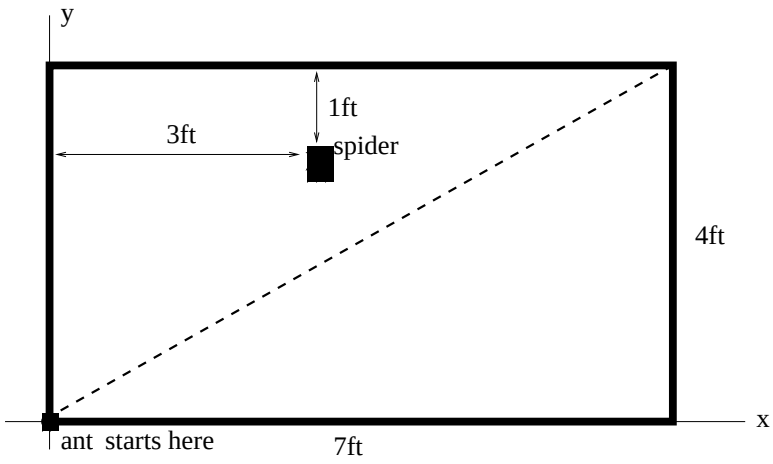
3. (3pts) What ticket price will maximize Ted's profit?

   The function $g(t)$ is quadratic, so we put into vertex form and find

   $$p = g(t) = -10(t - 4.5)^2 + g(4.5) = -10(t - 4.5)^2 + 162.5$$

   Conclude vertex is $(4.5, 162.5)$. Since leading coefficient of quadratic is negative, graph is downward opening parabola. Conclude vertex encodes maximum value of $g(t)$. Conclude Ted maximizes profit if sells tickets for \$4.50 each.

Problem 5. (24pts) A spider and an ant are located on a table top with dimensions and locations as pictured. The spider does not move, but the ant moves 2ft/minute along the dotted diagonal path. Impose coordinates as pictured.



(a) (2pts) Find the coordinates of the spider in the imposed coordinate system.

spider=(3,3).

(b) (5pts) If the ant has horizontal coordinate x, find the formula for a function $d(x)$ that gives the distance from the ant to the spider?

dotted line is $y = (4/7)x$, so ant position on line is $(x, (4/7)x)$, when ant x-coordinate is x. Apply distance formula to get

$$d(x) = \sqrt{(x-3)^2 + ((4/7)x - 3)^2}$$

(b) (11pts) Find the equation of a line through the spider perpendicular to the path of the ant. Where does this line intersect the path of the ant?

point is (3,3) and slope =-1/(slope of dotted line)=-7/4. By point slope formula, perpendicular line has equation

$$y = -7/4(x-3) + 3$$

Simulaneously solve this equation with dotted line equation to get intersection point:

$$-7/4(x-3) + 3 = (4/7)x$$

So, $x = 3.55$ and intersection point is $3.55, (4/7)3.55) = (3.55, 2.03)$.

(c) (6pts) WHEN does the ant reach the location closest to the spider?

Distance/Rate = Time

so,

$$\frac{\sqrt{3.55^2 + 2.03^2}}{2} = 2.04\text{minutes}.$$