1. [**12 points total**] Differentiate the following functions.

(a) [**4 points**] $g(t) = \sqrt{\dfrac{t^4 - 1}{1 - e^t}}$

$$g'(t) = \frac{1}{2} \cdot \left[\frac{t^4 - 1}{1 - e^t}\right]^{-1/2} \cdot \left[\frac{4t^3\,(1 - e^t) + (t^4 - 1)\,e^t}{(1 - e^t)^2}\right]$$

(b) [**4 points**] $y = \ln(\arcsin x + \arccos x)$

$$\arcsin x + \arccos x = \frac{\pi}{2} \quad \text{so} \quad y = \ln\frac{\pi}{2} \quad \text{and} \quad y' = 0$$

(c) [**4 points**] $f(x) = x^{e^x \cos(x)}$

$$f'(x) = x^{e^x \cos(x)} \cdot \left(e^x \cos(x) \ln(x) - e^x \sin(x) \ln(x) + e^x \cos(x) \cdot \frac{1}{x}\right)$$

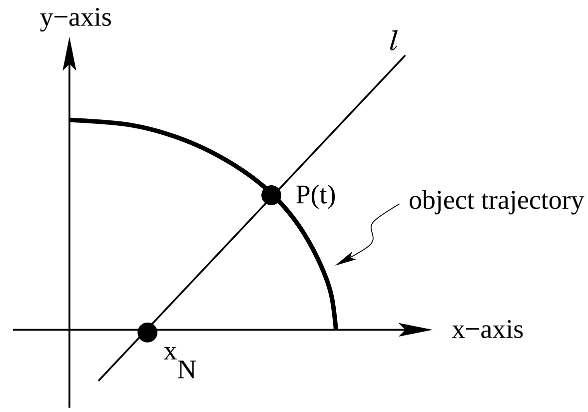
2. [**13 points total**] An object is moving in the xy-plane according to the parametric equations:

$$x(t) = 5\cos(\pi t)$$
$$y(t) = 4\sin(\pi t)$$

When $0 < t < \frac{1}{2}$, the location $P(t) = (x(t), y(t))$ of the object will be in the first quadrant, as pictured below. Let ℓ be the normal line to the trajectory at $P(t)$ and x_N the x-intercept of ℓ.

• The normal line ℓ is perpendicular to the tangent line through $P(t)$.

• Note that the chain rule says that $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.



(a) [**6 points**] Write the equation of the normal line ℓ through $P(t)$, assuming $0 < t < \frac{1}{2}$.

$$y - 4\sin(\pi t) = \frac{5}{4}\tan(\pi t)\,[x - 5\cos(\pi t)]$$

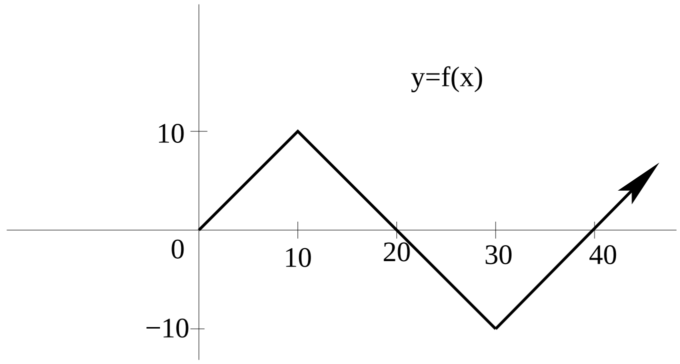
(b) [**4 points**] Find an expression for x_N as a function of t.

$$x_N = \frac{9}{5}\cos(\pi t)$$

(c) [**3 points**] Compute $\lim_{t\to 0^+} x_N =$

$$\lim_{t\to 0^+} \frac{9}{5}\cos(\pi t) = \frac{9}{5}$$

3. [**8 points total**] Here is the picture of the graph of a distance function $y = f(x)$. Distance is in feet, time is in seconds. Answer these questions; 1 point each, no partial credit.



(a) $f'(7) = 1$

(b) $\lim_{x \to 0} f(x + 20) = 0$

(c) $\lim_{x \to 0} \dfrac{f(x + 20)}{x} = -1$

(d) $\lim_{x \to 0} \dfrac{f(x + 20) - 20}{x - 20} = 1$

(e) The average velocity on the time interval [0,40] = 0

(f) The maximum velocity on the time interval [0,40] = 1

(g) $\lim_{x \to 10} f''(x) = 0$

(h) Let $g(x) = \dfrac{x}{x + 1}$ and $h(x) = f\,(g(x))$. Find $h'(10) = \dfrac{1}{121}$

4. [**12 points**] A particle travels in a straight line. After t seconds, the velocity is given by $v(t) = \dfrac{t}{3t+5}$ cm/sec. Compute a formula for the acceleration of the particle at time t.

Use only the limit definition of the derivative and not any differentiation formulas.

$$
\begin{aligned}
a(t) &= \frac{d}{dt}v(t) \\
&= \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{t+h}{3\,t+3\,h+5} - \frac{t}{3t+5}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{5\,h}{(3\,t+3\,h+5)\,(3\,t+5)}\right] \\
&= \lim_{h\to 0} \left[\frac{5}{(3\,t+3\,h+5)\,(3\,t+5)}\right] \\
&= \frac{5}{(3t+5)^2} \quad \text{by continuity}
\end{aligned}
$$

5. [**14 points**] A mass of clay of volume $\frac{4}{3}\pi\,\text{in}^3$ is formed into two spheres. How should the clay be divided to make the total surface area of the two spheres is
(a) a maximum?

(b) a minimum?

Note: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and its total surface area is $4\pi r^2$.

Let x and y be the radii of the two spheres.

Total volume is $\frac{4}{3}\pi x^3 + \frac{4}{3}\pi y^3 = \frac{4}{3}\pi$, *so* $x^3 + y^3 = 1$ *and* $y = \left(1 - x^3\right)^{1/3}$

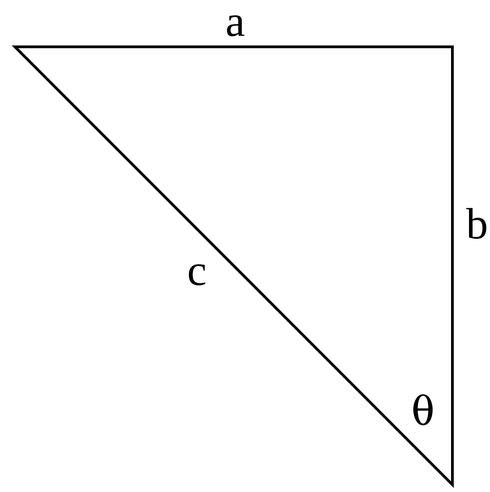
Find max/min of $S = 4\pi x^2 + 4\pi y^2 = 4\pi x^2 + 4\pi\left(1 - x^3\right)^{2/3}$ *on the interval* $0 \leq x \leq 1$

Compute $\frac{dS}{dx} = 8\pi x - \frac{8\pi x^2}{\sqrt[3]{1 - x^3}}$

Solve $\frac{dS}{dx} = 0$ *to get critical value* $x = \frac{1}{\sqrt[3]{2}}$

Check $x = 0,\ \frac{1}{\sqrt[3]{2}},\ 1.$

Conclude (a) two equal spheres give a max and (b) only one sphere gives a min.



6. [**13 points**] The hypotenuse of a right triangle is currently 10cm long and increasing at 0.1 cm/sec. The angle θ is $\frac{\pi}{4}$ and decreasing at .001 rad/sec.
How long are the other two sides and how fast are they changing?

$a^2 + b^2 = 100$ *and* $a = b$ *so* $a = b = \sqrt{50}$

$a = c\sin\theta$ *and* $b = c\cos\theta$

$$\frac{da}{dt} = \frac{dc}{dt}\cdot\sin\theta + c\cos\theta\cdot\frac{d\theta}{dt} = (0.1)\sin\left(\frac{\pi}{4}\right) + 10\cos\left(\frac{\pi}{4}\right)(-0.001)$$

$$\frac{db}{dt} = \frac{dc}{dt}\cdot\cos\theta - c\sin\theta\cdot\frac{d\theta}{dt} = (0.1)\cos\left(\frac{\pi}{4}\right) - 10\sin\left(\frac{\pi}{4}\right)(-0.001)$$

7. [**13 points total**] A problem of considerable importance in astronomy, arising in the study of planetary motion, is the determination of the "eccentric anomaly" E of the planet. It satisfies the equation:

$$E - c\sin(E) = \frac{2\pi t}{T}$$

where t is time in years, T is the period of the orbit and c is a constant called the eccentricity. Suppose $c = 0.9$, and $T = 10$ years.

(a) [**4 points**] Find $E'(t)$ when $E = \frac{\pi}{3}$.

Use implicit differentiation to get $E' - c\cos(E)\cdot E' = \frac{2\pi}{T}$

Plug in to get $E' = \frac{2\pi}{5.5}$ *when* $E = \frac{\pi}{3}$

(b) [**4 points**] When is $E = \frac{\pi}{3}$?

Plug in $\frac{\pi}{3} - c\sin\left(\frac{\pi}{3}\right) = \frac{2\pi t}{T}$

Use $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ *to get* $t = \frac{5}{\pi}\cdot\left(\frac{\pi}{3} - 0.9\cdot\frac{\sqrt{3}}{2}\right)$

(c) [**5 points**] Estimate E one month later.

Use the tangent line approximation $E - \frac{\pi}{3} \approx \frac{2\pi}{5.5}\cdot\left[t - \frac{5}{\pi}\cdot\left(\frac{\pi}{3} - 0.9\cdot\frac{\sqrt{3}}{2}\right)\right]$

Plug in $\Delta t = t - \frac{5}{\pi}\cdot\left(\frac{\pi}{3} - 0.9\cdot\frac{\sqrt{3}}{2}\right) = \frac{1}{12}$ *to get* $E \approx \frac{12\pi}{33}$

8. [**15 points total**] Consider the function $f(x) = (x-1)\,e^x$.

(a) [**1 point**] What are the x and y-intercepts of the graph of f?

x-intercept: $(1,0)$

y-intercept: $(0,-1)$

(b) [**3 points**] Using $f'(x)$, calculate the intervals in which f is increasing and decreasing.

$f'(x) = xe^x$

decreasing: $-\infty < x < 0$

increasing: $0 < x < \infty$

(c) [**2 points**] Calculate the minima and maxima of f.

local minimum at $x = 0$, the point is $(0,1)$

(d) [**3 points**] Using $f''(x)$ calculate the intervals in which f is concave up and concave down.

$f''(x) = (x+1)e^x$

concave down: $-\infty < x < -1$

concave up: $-1 < x < \infty$

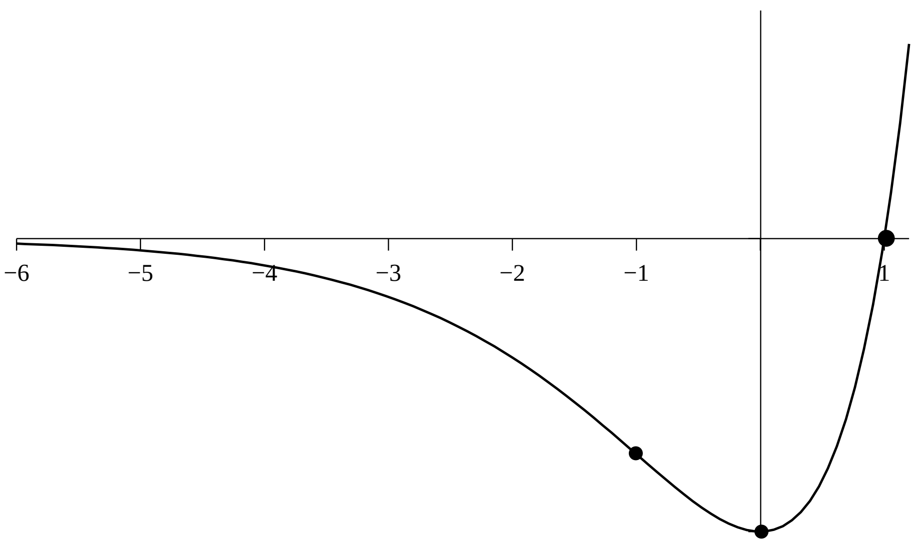
(e) [**2 points**] Calculate all horizontal asymptotes of f.

$\lim_{x\to\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = 0$

No horizontal asymptote as $x \to \infty$

Horizontal asymptote $y = 0$ *as* $x \to -\infty$

(f) [**4 points**] Using all the above information, make a careful sketch of the graph of f, labeling the axes and marking all the above information on the graph.



asymptote: $y = 0$ *(negative direction only)*

inflection point: $\left(-1, -\dfrac{2}{e}\right)$

y-intercept and local minimum: $(0, -1)$

x-intercept: $(1, 0)$