1. [**12 points total**] Calculate the derivatives of the following functions. You need not simplfy your answers.

(a) [**4 points**] $y = 5x^7 - \sin(3x) + e^2 + \ln x$.

$$y' = 35x^6 - 3\cos(3x) + 0 + \frac{1}{x}$$

(b) [**4 points**] $f(r) = \left(1+\sqrt{r}\right)^3 \cdot \left(1 - \frac{1}{\sqrt{r}}\right)^3$

$$f'(r) = 3(1+\sqrt{r})^2\left(\frac{1}{2\sqrt{r}}\right)\left(1-\frac{1}{\sqrt{r}}\right)^3 + (1+\sqrt{r})^3 \cdot 3\left(1-\frac{1}{\sqrt{r}}\right)^2\left(\frac{1}{2r^{3/2}}\right)$$

OR

if they simplified first

$$f(r) = (r^{1/2} - r^{-1/2})^3$$
$$f'(r) = \frac{3}{2}(r^{1/2} - r^{-1/2})^2 \cdot (r^{-1/2} + r^{-3/2})$$

(c) [**4 points**] $g(t) = (\ln t)^{\ln t}$

$$u = (\ln t)^{\ln t}$$

$$\ln u = (\ln t)(\ln(\ln t))$$

$$\frac{u'}{u} = \frac{1}{t} \cdot (\ln(\ln t)) + (\ln t) \cdot \frac{1}{t(\ln t)} = \frac{\ln(\ln t) + 1}{t}$$

$$g'(t) = \frac{\ln(\ln t) + 1}{t} \cdot (\ln t)^{\ln t}$$

2. [**10 points total**] Compute the following limits. You must show your work or adequately explain your answers. No credit will be given for an unsupported answer.

(a) [**5 points**] $\lim_{x\to 2} \frac{\frac{2}{x^2}-\frac{1}{2}}{x-2}$

$$\begin{aligned}\lim_{x\to 2} \frac{\frac{2}{x^2}-\frac{1}{2}}{x-2} &= \lim_{x\to 2} \frac{4-x^2}{2x^2(x-2)} \\ &= \lim_{x\to 2} \frac{-(x+2)}{2x^2} = -\frac{1}{2}\end{aligned}$$

(b) [**5 points**] $\lim_{\theta\to 0} \frac{\sin(\theta^2)}{1-\cos(\theta)}$

This is the $\frac{0}{0}$ *case of l'Hôpital's Rule.*

$$\begin{aligned}\lim_{\theta\to 0} \frac{\sin(\theta^2)}{1-\cos(\theta)} &= \lim_{\theta\to 0} \frac{2\theta\cos(\theta^2)}{\sin(\theta)} && \textit{by l'Hôpital's Rule, still } \frac{0}{0} \textit{ case} \\ &= \lim_{\theta\to 0} \frac{2\cos(\theta^2)-4\theta^2\sin(\theta^2)}{\cos(\theta)} && \textit{by l'Hôpital's Rule} \\ &= 2 \quad \textit{using continuity}\end{aligned}$$

3. [**12 points total**] Let $f(x) = e^{2x}\cos(2x)$ on the domain $D = [-1, 2]$.

(a) [**3 points**] Find the subintervals of D on which f is increasing, and the subintervals on which f is decreasing.

To find the intervals on which f is increasing or decreasing, we compute the derivative f' and determine the intervals on which it is positive or negative. To compute the derivative we use the product rule and chain rule to obtain

$$f'(x) = e^{2x} \cdot -\sin(2x) \cdot 2 + \cos(2x) \cdot e^{2x} \cdot 2 = 2e^{2x}(\cos(2x) - \sin(2x)).$$

Since e^{2x} is never zero, the derivative can be zero only where $\cos(2x) - \sin(2x) = 0$. Now, $\cos(2x) = \sin(2x)$ when $x = \frac{\pi}{8}$ or $x = \frac{5\pi}{8}$ (in the given interval). Therefore, we must consider three intervals $[-1, \frac{\pi}{8}]$, $[\frac{\pi}{8}, \frac{5\pi}{8}]$ and $[\frac{5\pi}{8}, 2]$. Evaluating at 0, we see that on the first interval $f'(x)$ is positive, evaluating at $x = \frac{\pi}{4}$, we see that $f'(x)$ is negative on the second interval and evaluating at 2, we see that $f'(x)$ is positive on the third interval.

(b) [**3 points**] Find the subintervals of D on which f is concave up and concave down.

To study the concavity properties of f, we need to determine where the second derivative is positive or negative. To do this, we find the points in the interval D where $f''(x) = 0$. Taking the derivative of f' using the product rule and chain rule again

$$f''(x) = 2e^{2x} \cdot (-\sin(2x) \cdot 2 - \cos(2x) \cdot 2) + (\cos(2x) - \sin(2x)) \cdot 2e^{2x} \cdot 2$$

Simplifying this, we obtain $f''(x) = 4e^{2x}(-\sin(2x) - \cos(2x) + \cos(2x) - \sin(2x)) = -8e^{2x}\sin(2x)$. Now, since $-8e^{2x}$ is never 0, the only places the second derivative can be zero are the values of x where $\sin(2x) = 0$. This can happen only if $2x$ is a multiple of π, or in other words where $x = 0$ or $x = \frac{\pi}{2}$. Plugging in values shows that $f''(x)$ is positive on $[-1, 0]$, negative on $[0, \frac{\pi}{2}]$ and positive on $[\frac{\pi}{2}, 2]$. Therefore, f is concave up on the first and third intervals and concave down on the second interval.

(c) [**2 points**] What are the local minima and maxima?

We've already shown that f has two critical points, namely $x = \frac{\pi}{8}$ and $x = \frac{5\pi}{8}$. The second derivative test shows that $x = \frac{\pi}{8}$ is local maximum and $x = \frac{5\pi}{8}$ is a local minimum. Furthermore, the endpoints are local maxima and minima.

(d) [**2 points**] What are the global minimum and maximum?

By explicit evaluation, we see that the global minimum occurs at $x = \frac{5\pi}{8}$ and the global maximum occurs at $x = \frac{\pi}{8}$. The same conclusion also follows from the second derivative test.

(e) [**2 points**] If you used the tangent line approximation to f at $x = 1.0$ to approximate $f(1.1)$, will it give an underestimate or an overestimate. Why?

Since at $x = 1$, the second derivative is negative, the function is concave down. Therefore, the tangent line approximation would give an over-estimate.

4. [**10 points total**] Let c be a constant and define $f(x)$ by the piecewise formula

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ c & \text{if } x = 0. \end{cases}$$

(a) [**2 points**] Find a value of c that makes $f(x)$ continuous on $(-\infty, \infty)$. Use this value of c for the rest of the problem.

$$c = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

If follows from the Squeeze Theorem,

because

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

and

$$\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$$

(b) [**6 points**] Calculate $f'(0)$. **Use the definition of derivative.** (The differentiation formula won't work here!)

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x \to 0} x \sin \frac{1}{x} = 0 \end{aligned}$$

Again, it follows from the Squeeze Theorem,

because

$$-|x| \leq x \sin \frac{1}{x} \leq |x|$$

and

$$\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|.$$

(c) [**2 points**] Is the function $f'(x)$ continuous on $(-\infty, \infty)$?

No.

Because

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

and $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$ *does not exist.*

5. [**12 points total**] Gravitational force always pulls along lines perpendicular to *equi-potential curves*. If two equal masses are located at points, P=(0,4) and Q=(9,0), then the equi-potential curves are described by the condition:

$$\frac{1}{\text{distance from } (x,y) \text{ to P}} + \frac{1}{\text{distance from } (x,y) \text{ to Q}} = \text{Constant.}$$

(a) [**3 points**] Write an implicit equation for the equi-potential curve passing through $(3,8)$.

$$\frac{1}{\sqrt{x^2+(y-4)^2}} + \frac{1}{\sqrt{(x-9)^2+y^2}} = \frac{3}{10}.$$

(b) [**4 points**] What it the slope of the tangent line to this equi-potential curve at $(3,8)$? Please show your work clearly.

$$-(x+(y-4)y')(x^2+(y-4)^2)^{-3/2} - (x-9+yy')((x-9)^2+y^2)^{-3/2} = 0,$$

so that at $(x,y)=(3,8)$ we have

$$-(3+4y')/125 - (-6+8y')/1000 = 0$$

multiplying through by 1000 we get $-18-40y'=0$, so

$$y' = -18/40 = -9/20 = -0.45.$$

(c) [**3 points**] What is the equation of the line through $(3,8)$ perpendicular to the equi-potential curve? (This line gives the direction of the gravitational force).

The slope is the negative reciprocal of the answer to (b), which is $20/9$, and the point-slope formula gives

$$y = \frac{20}{9}x + \frac{4}{3}.$$

(d) [**2 points**] What angle does the line in part (c) make with the x-axis? (Remember to give units in your answer.)

$$tan^{-1}(20/9) \approx 65.77^\circ;$$

6. [**12 points total**] A vat contains a salt water solution. Fresh water enters the vat at a constant rate, and the thoroughly mixed solution exits the vat at the same rate. As a result, the concentration of salt is a (decaying) exponential function of time.

Let $y = f(t)$ be the concentration of salt in milligrams per liter at time t hours. Suppose that at time $t = 0$ the concentration is 25.000 mg/ℓ, and two hours later it has been reduced to 20.000 mg/ℓ. Carry out all your work to five significant figures of accuracy.

(a) [**4 points**] Find a formula for y as a function of t. Please show your work clearly.

$$y = f(t) = 25e^{-bt},$$

where

$$20 = f(2) = 25e^{-2b},$$

so that

$$b = \frac{1}{2}\ln(25/20) = 0.11157.$$

(b) [**4 points**] Find an equation for the tangent line to this function at the point when $t = 2$.

$$f'(t) = -25be^{-bt} = -bf(t),$$

so

$$f'(2) = 20 \cdot (-0.11157) = -2.2314,$$

and so the equation of the tangent line is

$$y = -2.2314t + 24.463.$$

(c) [**4 points**] Using part (b), estimate the concentration of salt after 2 hours 15 minutes. How far is this answer from the exact value obtained using the formula in part (a)?

$$f\left(2\frac{1}{4}\right) \approx f(2) + f'(2) \cdot \left(\frac{1}{4}\right) = 19.442;$$

the exact value is

$$25e^{-0.11157 \cdot 2.25} = 19.450.$$

7. [**10 points total**] A highway patrol officer's radar unit is parked behind a bulletin board 200 ft from a long straight stretch of highway US5. Down the highway, 200 feet from the point on the highway closest to the officer, is an emergency call box. A truck passes the call box and, at that moment, the radar unit indicates that the distance between the officer and the truck is increasing at a rate of 45 miles per hour. The posted speed limit is 55 miles per hour. Calculate the speed of the truck. Should the officer apprehend the driver of the truck for speeding?

Let us introduce variables as in Figure 1.

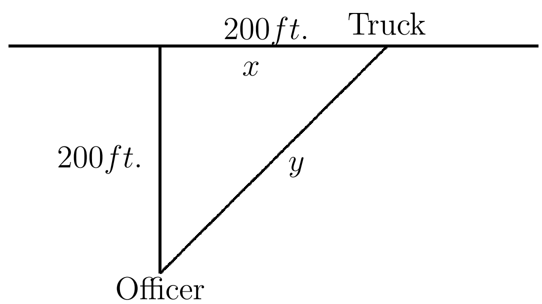

Figure 1: Speeding

We want to compute $\frac{dx}{dt}$ *and we know* $\frac{dy}{dt}$.
We need to be careful with units though: note that $1mi = 5280ft$.
The Pythagorean theorem gives the relation

$$x^2 + \left(\frac{200}{5280}\right)^2 = y^2.$$

The triangle in figure 1 *is an isosceles right triangle and hence* $y = 200\sqrt{2}ft$ *or, in terms of miles,*

$$y = \frac{200\sqrt{2}}{5280}mi.$$

Differentiating the relation, we obtain

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}.$$

Substituting $x = \frac{200}{5280}mi$, $y = \frac{200\sqrt{2}}{5280}mi$, *and* $\frac{dy}{dt} = 45mi/hr$, *we see that*

$$\frac{dx}{dt} = 45\sqrt{2}mi/hr.$$

Now $\sqrt{2} > 1.4$, *so*

$$\frac{dx}{dt} > 63mph;$$

the officer should apprehend the driver of the truck.

8. [**10 points total**] A rectangular poster is to have an area of 2700 cm^2 with 3-cm margins at the bottom and sides and a 5-cm margin at the top. What dimensions will give the largest printed area (the area inside the margins)? Please show your work, and justify that your solution truly is a maximum.

If x is the width, then $2700/x$ is the height, and you want to maximize

$$\begin{aligned} y &= (x-6)(2700/x-8) \\ &= 2748 - 8x - 16200/x. \end{aligned}$$

Since

$$y' = -8 + 16200/x^2,$$

solving $y' = 0$ gives

$$x = \sqrt{16200/8} = \sqrt{2025} = 45,$$

so the dimensions are 45 cm by 60 cm. Since

$$y'' = -32400/x^3$$

is negative for any positive x, the second derivative test shows that it's a maximum.

9. [**12 points total**] Consider the function $g(x)$ whose graph is given below. The domain of the function is $[-2, 4]$.

(a) [**2 points**] Is $g(x)$ continuous? Explain briefly why or why not.

No, $g(x)$ is not continuous as it has a jump discontinuity at $x = \frac{1}{2}$: the limits as x approaches $\frac{1}{2}$ from the right and left do not agree.

(b) [**2 points**] Is the derivative of $g(x)$ continuous? If not, explain where points of discontinuity in the graph of the derivative occur.

No, the derivative is not continous. As the function is discontinuous at $x = \frac{1}{2}$, the derivative is not defined there. Furthermore, the function has a "corner" at $x \sim \frac{9}{4}$ (the slopes of the tangent lines from either side are of opposite signs).

(c) [**2 points**] On (approximately) which interval(s) is the derivative positive? Where does the derivative take the value zero?

The derivative appears to take the value 0 at $x = 0$ and perhaps at $x = -\frac{3}{2}$ as the tangent lines are horizontal there. The derivative appears to be positive on the intervals $(-\frac{3}{2}, 0)$ and $(\frac{1}{2}, \frac{9}{4})$.

(d) [**2 points**] On what interval(s) is the *second* derivative negative?

The second derivative is negative when f is concave down. This appears to be the case on the interval $[-\frac{3}{4}, \frac{1}{2})$.

(e) [**4 points**] Sketch a graph of the *derivative* of $g(x)$, making sure to indicate all the features you described in the previous parts.