1. [**15 points total**] Find dy/dx for the following.

(a) [**5 points**] $y = \sqrt{x} - \dfrac{1}{\sqrt{3x-5}}$

$$y' = \frac{1}{2\sqrt{x}} + \frac{3}{2(3x-5)^{3/2}}$$

(b) [**5 points**] $y = \dfrac{\sin(x^2)}{\cos^3(2x)}$

$$y' = \frac{(2x\cos(x^2))(\cos^3(2x)) - (\sin(x^2))(3\cos^2(2x)\sin(2x)2)}{\cos^6(2x)}$$

(c) [**5 points**] $y = (x^2+1)^{(x+1)}$

$$\ln y = \ln(x^2+1)^{(x+1)} = (x+1)\ln(x^2+1)$$

$$\frac{y'}{y} = \ln(x^2+1) + (x+1)\frac{2x}{x^2+1}$$

$$y' = (\ln(x^2+1) + \frac{2x(x+1)}{x^2+1})(x^2+1)^{(x+1)}$$

2. [**10 points total**] Evaluate the following limits.

(a) [**5 points**] $\lim_{t\to\infty} \frac{t^2+1}{t\ln t}$

$$\lim_{t\to\infty} \frac{t^2+1}{t\ln t} = \lim_{t\to\infty} \frac{2t}{\ln t + 1} = \lim_{t\to\infty} \frac{2}{1/t} = \lim_{t\to\infty} 2t = \infty$$

(b) [**5 points**] $\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{4x^2-8}}$

$$\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{4x^2-8}} = \lim_{x\to\infty} \frac{\sqrt{1-\frac{1}{x^2}}}{\sqrt{4-\frac{8}{x^2}}} = \lim_{x\to\infty} \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

3. [**12 points total**] A tennis ball is dropped from a height of 10 feet at time $t = 0$ seconds. It bounces up and down for the next 2π seconds according to the following function:

$$s(t) = 10\, e^{-2t} \sin^2\left(t + \frac{\pi}{2}\right) \qquad 0 \leq t \leq 2\pi.$$

Where $s(t)$ is the distance of the ball from the ground.

(a) [**4 points**] Find **all** the times when the velocity of the ball is zero.

$$v(t) = \frac{ds}{dt} = -20e^{-2t} \cos(t)\left[\cos(t) + \sin(t)\right]$$

$$v(t) = 0 \quad \Rightarrow \quad \cos(t) = 0 \text{ or } \cos(t) + \sin(t) = 0 \quad \Rightarrow \quad t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

(b) [**6 points**] Find all values of t for which $s(t)$ is a local maximum and determine the value of $s(t)$ at those points.

$$s''(t) = 20e^{-2t}\left[1 + 4\sin(t)\cos(t)\right]$$

$$s''(\pi/2) = 20e^{-\pi} > 0 \text{ so } t = \pi/2 \text{ is a local minimum}$$

$$s''(3\pi/2) = 20e^{-3\pi} > 0 \text{ so } t = 3\pi/2 \text{ is a local minimum}$$

$$s''(3\pi/4) = -20e^{-3\pi/2} < 0 \text{ so } t = 3\pi/4 \text{ is a local maximum}$$

$$s''(7\pi/4) = -20e^{-7\pi/2} < 0 \text{ so } t = 7\pi/4 \text{ is a local maximum}$$

$$s(3\pi/4) = 5e^{-3\pi/2}$$

$$s(7\pi/4) = 5e^{-7\pi/2}$$

(c) [**2 points**] Determine the global maximum of $s(t)$.

We need to check endpoints and local maxima.

$$\begin{aligned} s(0) &= 10, \quad s(2\pi) = 10e^{-4\pi} \\ s(0) &> s(3\pi/4) > s(7\pi/4) > s(2\pi) \Rightarrow \textit{ global maximum is at } t = 0,\ s(0) = 10 \end{aligned}$$

4. [**10 points**] Find the derivative of the function $f(x) = x + \frac{2}{x}$ using the definition of the derivative. **Do not use any differentiation formulas.**

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\left[x+h+\frac{2}{x+h}-x-\frac{2}{x}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\cdot\left[h+\frac{-2h}{x(x+h)}\right] \\ &= \lim_{h\to 0} \left[1-\frac{2}{x(x+h)}\right] \\ &= 1-\frac{2}{x^2} \quad \textit{by continuity} \end{aligned}$$

5. [**10 points**] Use linear approximation to estimate the y value of a point $\left(\frac{99\pi}{100}, y\right)$ on the curve $y = \sin(y+x)$ near $(\pi, 0)$.

1. Linear Function:

$\frac{d}{dx}(y) = \frac{d}{dx}\sin(y+x)$

$\frac{dy}{dx} = \cos(y+x)\left(\frac{dy}{dx}+1\right)$

$\frac{dy}{dx} = \cos\pi\left(\frac{dy}{dx}+1\right) = -\left(\frac{dy}{dx}+1\right)$

so $\left.\frac{dy}{dx}\right|_{(\pi,0)} = -\frac{1}{2}$

$y - 0 = -\frac{1}{2}(x-\pi)$,

The linear function is $y = -\frac{1}{2}(x-\pi)$

2. Approximation:

$y\left(\frac{99}{100}\pi\right) \approx -\frac{1}{2}\left(\frac{99}{100}\pi - \pi\right) = \frac{\pi}{200}.$

6. [**15 points total**] Consider the function $P(x) = x^2\, e^{-x^2+2}$. The domain is all real numbers.

(a) [**3 points**] Calculate the intervals in which $P(x)$ is increasing and decreasing.

$P'(x) = -2\,x\ (x-1)\ (x+1)\ e^{-x^2+2}$

$P(x)$ is increasing when $P'(x) > 0$, that is $x \in (-\infty, -1) \cup (0, 1)$.

$P(x)$ is increasing when $P'(x) < 0$, that is $x \in (-1, 0) \cup (1, \infty)$.

(b) [**3 points**] Find all local extrema for $P(x)$ and justify your answers.

$P'(x) = -2\,x\ (x-1)\ (x+1)\ e^{-x^2+2} = 0$ *when* $x = 0, -1, 1$

Using (a) we see that $(0, 0)$ is a relative minimum and $(-1, e), (1, e)$ are relative maxima.

(c) [**3 points**] Calculate the intervals where $P(x)$ is concave up and concave down.

$P''(x) = 2\left(1 - 5\,x^2 + 2\,x^4\right)\, e^{-x^2+2}$

$$
\begin{aligned}
1 - 5\,x^2 + 2\,x^4 &= \frac{1}{8}\left(4\,x^2 - 5 + \sqrt{17}\right)\left(4\,x^2 - 5 - \sqrt{17}\right) \\
&= \frac{1}{8}\left(2\,x - \sqrt{-\sqrt{17}+5}\right)\left(2\,x + \sqrt{-\sqrt{17}+5}\right)\left(2\,x - \sqrt{\sqrt{17}+5}\right)\left(2\,x + \sqrt{\sqrt{17}+5}\right)
\end{aligned}
$$

$P(x)$ is concave up when $x \in (-\infty, -1.51) \cup (-0.468, 0.468) \cup (1.51, \infty)$

$P(x)$ is concave down when $x \in (-1.51, -0.468) \cup (0.468, 1.51)$

(d) [**2 points**] Find all horizontal asymptotes of $P(x)$.

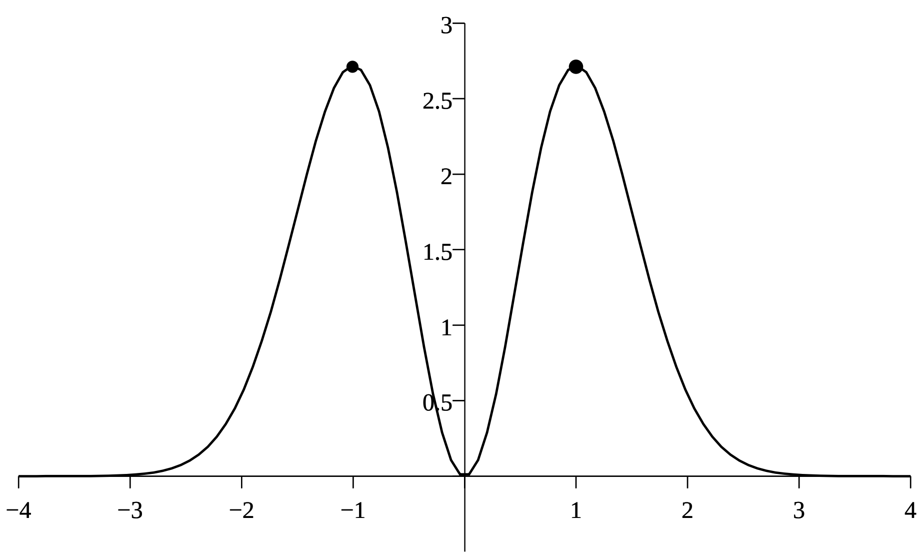
$$\begin{aligned}\lim_{x\to\infty} P(x) &= \lim_{x\to\infty} \frac{x^2}{e^{x^2-2}} \\ &= \lim_{x\to\infty} \frac{2x}{2x\, e^{x^2-2}} \quad \textit{by L'Hôpital's Rule} \\ &= \lim_{x\to\infty} \frac{1}{e^{x^2-2}} = 0\end{aligned}$$

Thus $y = 0$ is a horizontal asymptote as $x \to \infty$

$$\begin{aligned}\lim_{x\to-\infty} P(x) &= \lim_{x\to-\infty} \frac{x^2}{e^{x^2-2}} \\ &= \lim_{x\to-\infty} \frac{2x}{2x\, e^{x^2-2}} \quad \textit{by L'Hôpital's Rule} \\ &= \lim_{x\to-\infty} \frac{1}{e^{x^2-2}} = 0\end{aligned}$$

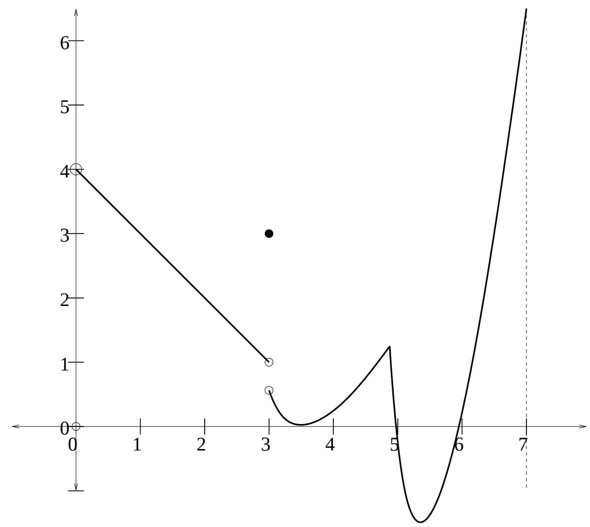
Thus $y = 0$ is a horizontal asymptote as $x \to -\infty$

(e) [**4 points**] Sketch the graph of $P(x)$ below, labeling your extrema and indicating any asymptotes and x-intercepts and y-intercepts.



The marked points are the relative (and global) maxima. The origin is the global minimum, and the only x and y-intercept. The x-axis is a horizontal asymptote (in both directions).

7. [**16 points total**] To the right is a sketch of the graph of $y = f(x)$ which has domain $(0, 7)$. Note that $x = 7$ is a vertical asymptote.



Take also $g(x) = f(f(x^2 + 3))$.

Estimate the following values; 2 points each, no partial credit.

(a) the domain of $f'(x)$

$$(0, 3) \cup (3, \text{about } 4.9) \cup (\text{about } 4.9, 7)$$

(b) the domain of $g(x)$

Let $a = \sqrt{0.5}$, $b = \sqrt{2}$, $c = \sqrt{2.95}$ *and* $d = \sqrt{3.9}$.

The domain is $(-a, a) \cup (-b, -a) \cup (a, b) \cup (-c, -d) \cup (c, d)$

(c) $\lim_{a \to 3^-} f(a) = 1$

(d) $\lim_{a \to 3^-} f'(a) = -1$

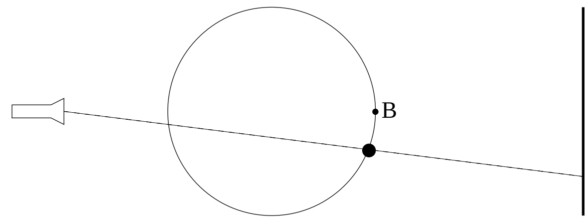
(e) $f''(1) = 0$

(f) $f(f(f(3))) = 3$

(g) $\lim_{a \to 7^-} f'(a) = \infty$ (or "Does not exist")

(h) $g'(1) = -2f'(4) \approx \dfrac{-4}{2.3}$

8. [**12 points**] Starting at point B, an object travels with constant speed 7 cm per second counterclockwise around a circle of radius 10 cm. A light is positioned 20 cm from the center of the circle, directly across from point B. A wall is positioned 30 cm from the center of the circle directly opposite the light. Find the velocity of the shadow cast by the object on the wall as a function of time.



Choose a coordinate system with the origin at the center of the circle and the y-axis parallel to the wall.

The angular speed is $\dfrac{2\pi \ rad}{20\pi \ cm} \cdot 7\dfrac{cm}{sec} = \dfrac{7 \ rad}{10 \ sec}$

The initial angle is $\theta = 0$

The parametric equations of the object are

$$\begin{aligned} x(t) &= 10\cos\left(\frac{7}{10}t\right) \\ y(t) &= 10\sin\left(\frac{7}{10}t\right) \end{aligned}$$

Let s represent the y-coordinate of the shadow on the wall. Let (x, y) be the coordinates of the object.

By similar triangles, $\dfrac{s}{50} = \dfrac{y}{x+20}$

Thus $s(t) = \dfrac{50\sin\left(\frac{7}{10}t\right)}{\cos\left(\frac{7}{10}t\right)+2}$

$$\begin{aligned} v(t) &= \frac{ds}{dt} \\ &= 35\,\frac{\cos^2\left(\frac{7}{10}t\right)+\sin^2\left(\frac{7}{10}t\right)+2\cos\left(\frac{7}{10}t\right)}{\left[\cos\left(\frac{7}{10}t\right)+2\right]^2} \\ &= 35\,\frac{1+2\cos\left(\frac{7}{10}t\right)}{\left[\cos\left(\frac{7}{10}t\right)+2\right]^2}\ \frac{cm}{sec} \end{aligned}$$