1 (10 points) Determine if the following limits exist. If they exist, compute them (including the case when the limit is $\pm\infty$. Justify your answers.

(a) (3 points) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

Note that you cannot simply substitute $x = 0$, because you'll get $0/0$. So you have to change the form of the expression in the limit somehow. We'd like to get rid of the square root in the numerator, so we use a trick that appeared on at least one homework problem: Multiply the top and the bottom of the fraction by $\sqrt{4+x}+2$ to get

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \lim_{x\to 0} \frac{4+x-4}{x\sqrt{4+x}+2} = \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}.$$

(b) (3 points) $\lim_{x\to\infty} \frac{5x^4-2x^2+1}{7x^4+x^3-3}$

Note that both the numerator and the denominator go to infinity as $x \to \infty$. So again, you have to change the form of the expression somehow. In this case there is a standard way: Divide the top and the bottom by the highest power of x that occurs, namely x^4 in this case. This yields

$$\lim_{x\to\infty} \frac{5x^4-2x^2+1}{7x^4+x^3-3} = \lim_{x\to\infty} \frac{5-2x^{-2}+x^{-4}}{7+x^{-1}-3x^{-4}} = \frac{5}{7}.$$

(c) (4 points) $\lim_{x\to\frac{\pi}{2}^+} e^{\tan x}$ and $\lim_{x\to\frac{\pi}{2}^-} e^{\tan x}$

As x approaches $\pi/2$ from the right, $\tan x$ goes to $-\infty$. Therefore

$$\lim_{x\to\frac{\pi}{2}^+} e^{\tan x} = 0.$$

As x approaches $\pi/2$ from the left, $\tan x$ goes to $+\infty$. Therefore

$$\lim_{x\to\frac{\pi}{2}^+} e^{\tan x} = \infty,$$

and the limit does not exist (even though it's equal to infinity!—strange terminology, but that's the way it is).

2 (10 points) **Do not use differentiation rules on this problem. Use limits where appropriate.** According to Boyle's law, if the temperature of a confined gas is held fixed, then $V = \frac{c}{P}$, where V is volume (in cubic inches), P is pressure (in pounds per square inch), and c is a constant. Assume that $c = 500$.

a)(2 points) Find the average rate of change of V as P increases from 10 to 20.

$\frac{\Delta V}{\Delta P} = \frac{(500/20)-(500/10)}{20-10} = -2.5$, where the units are cubic inches divided by pounds per square inch.

b)(8 points) Find the instantaneous rate of change of V with respect to P when $P = 20$.

We need to compute $\frac{dV}{dP}$ and then evaluate it at $P = 20$.

$$\frac{dV}{dP} = \lim_{h\to 0} \frac{(c/(P+h) - c/P}{h} = \lim_{h\to 0} \frac{cP - c(P+h)}{hP(P+h)} = \lim_{h\to 0} \frac{-c}{P(P+h)} = -c/P^2.$$

Evaluating at $P = 20$ we get $-500/400 = -1.25$. (same units)

3 (10 points) **Do not use differentiation rules on this problem. Use limits where appropriate.** Find the equation of the tangent line to $y = 2x^3 + 1$ at the point $(2, 17)$.

The slope of the tangent line is the value of the derivative at $x = 2$. So we first compute the derivative. The difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{2(x+h)^3 + 1 - (2x^3 + 1)}{h} = \frac{2(x^3 + 3x^2h + 3xh^2 + x^3) - 2x^3}{h} = 2(3x^2 + 3xh + h^2).$$

Now let $h \to 0$ to get $\frac{dy}{dx} = 6x^2$. Plugging in $x = 2$, we find the slope of the tangent line is 24. In point-slope form the desired equation is then $y - 17 = 24(x - 2)$.

4 (10 points) **You may and should use the differentiation rules on this problem, where appropriate.** Suppose $f(x) = 4x^3e^x + 5$. Find all values of a (if any) such that the graph of f has a horizontal tangent line at the point $(a, f(a))$.

The tangent line at $(a, f(a))$ is horizontal precisely when $f'(a) = 0$. So we first compute the derivative using the product rule:

$$f'(x) = 12x^2e^x + 4x^3e^x = 4e^x(3x^2 + x^3).$$

Since e^x is never zero, the horizontal tangents will occur when $3x^2 + x^3 = 0$, i.e., when $x = 0$ or $x = -3$.

5 (10 points) **You may and should use the differentiation rules on this problem, where appropriate.** A spherical balloon is expanding or shrinking according to the formula $r(t) = \dfrac{t}{1+t^2}$ where r is the radius of the balloon (in meters) and t is time (in minutes).

a)(8 points) For what values of t is the balloon expanding? For what values of t is it shrinking?

Expansion corresponds to r increasing; i.e. $dr/dt > 0$. Shrinking corresponds to r decreasing; i.e. $dr/dt < 0$. So we first compute the derivative using the quotient rule:

$$\frac{dr}{dt} = \frac{1\cdot(1+t^2) - 2t\cdot t}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}.$$

Since the denominator is always positive, the sign of dr/dt is the same as the sign of the numerator. Thus the balloon is expanding when $1 - t^2 > 0$; i.e., $|t| < 1$. It is shrinking when $|t| < 1$. Here I'm thinking of time as positive, so it expands when $t < 1$ (minutes) and shrinks when $t > 1$ (minutes). But it's fine if you allowed for negative times as well.

b)(2 points) What happens as $t \to \infty$? Justify your answer by computing the appropriate limit.

The radius of the balloon shrinks to zero. In precise mathematical terms, we show this by the computation

$$\lim_{t\to\infty} \frac{t}{1+t^2} = \lim_{t\to\infty} \frac{t/t^2}{1/t^2 + t^2/t^2} = \lim_{t\to\infty} \frac{1/t}{1/t^2+1} = \frac{0}{1} = 0.$$

Woo hoo! Yee ha! We are done!!