Solutions to Midterm 1 - Winter 2003

Problem 1

a)

Equations for the crow: $\begin{cases} x = x_1(t) = 2t - 1 \\ y = y_1(t) = 4t + 5 \end{cases}$

Solving for t in the first equation: $t = \frac{x+1}{2}$. Substituting in the second equation, $y = 4\left(\frac{x+1}{2}\right) + 5 = 2x + 2 + 5 = 2x + 7$. Since $t \geq 0$ then $x \geq -1$.

<u>Solution</u>: $y = 2x + 7$ when $x \geq -1$, a line.

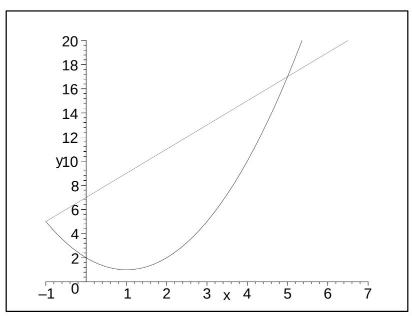
Equations for the pigeon: $\begin{cases} x = x_2(t) = t - 1 \\ y = y_2(t) = (t-2)^2 + 1 \end{cases}$

Solving for t in the first equation, $t = x + 1$, and substituting in the second equation $y = (x + 1 - 2)^2 + 1 = (x - 1)^2 + 1$. Since $t \geq 0$ then $x \geq -1$.

<u>Solution</u>: $y = (x - 1)^2 + 1$ for $x \geq -1$, a parabola.

b)

The line starts at $(-1, 5)$ and has slope 2. The parabola starts at $(-1, 5)$ and its vertex is at $(1, 1)$.



c)

Time at which the crow gets to $(5, 17)$:

$$\begin{aligned} 2t - 1 &= 5 \\ 4t + 5 &= 17. \end{aligned}$$

Solve for t in any of the two equations, and get $t = 3$.

Time at which the pigeon gets to $(5, 17)$:

$$\begin{aligned} t - 1 &= 5 \\ (t-2)^2 + 1 &= 17. \end{aligned}$$

Solve for t in any of the two equations, and get $t = 6$.

Solution: The crow and the pigeon do not collide, since the crow gets to $(5, 17)$ before the pigeon does.

Problem 2

a)

If C is the concentration of carbon 14 then, $C(t) = ab^t$.

$$\begin{aligned} C(0) &= ab^0 = a = 2, \\ C(-3) &= ab^{-3} = 54. \end{aligned}$$

$2b^{-3} = 54$; $b^{-3} = 27$; $b^3 = 1/27$; $b = (1/27)^{1/3} = 1/3$.

Solution: $C(t) = 2(1/3)^t$.

b)

Solution: $C(1) = 2(1/3) = 2/3$

c)

$10 = 2(1/3)^t$; $(1/3)^t = 5$; $\quad 3^t = 1/5$; $\quad t = \frac{\ln(1/5)}{\ln(3)} = -1.465$.

Solution: Roughly, 1,500 years ago.

Problem 3

a)

$$\frac{d(5) - d(0)}{5 - 0} = \frac{-(5)^2 + 10(5) + 4 - (-(0)^2 + 10(0) + 4)}{5 - 0} = \frac{-25 + 50}{5} = 5.$$

Solution: 5 feet/sec.

b)

$$\begin{aligned} \frac{d(t+h) - d(t)}{(t+h) - t} &= \frac{-(t+h)^2 + 10(t+h) + 4 - (-(t)^2 + 10(t) + 4)}{h} \\ &= \frac{-t^2 - 2th - h^2 + 10t + 10h + 4 + t^2 - 10t - 4}{h} \\ &= \frac{-2th - h^2 + 10h}{h} = -2t - h + 10. \end{aligned}$$

Therefore, the instantaneous velocity is,

$$\lim_{h \to 0} \frac{d(t+h) - d(t)}{(t+h) - t} = \lim_{h \to 0} -2t - h + 10 = -2t + 10.$$

Solution: $-2t + 10$ feet/sec.

c)

$-2t + 10 = 0$; $t = 10/2 = 5$.

Solution: $t = 5$ seconds.

d)

The ball is at its highest position when it stops, that is, when the instantaneous velocity is zero. By part b), $t = 5$.

Solution: $t = 5$ seconds.

(Another way of doing it. The ball is at its highest position at the vertex of the parabola (since it is a downwards convex parabola). The vertex is at $t = -\frac{10}{-2} = 5$.

Solution: $t = 5$ seconds.)

Problem 4

a)

For $x \leq 3$, $f(x)$ is not defined when the denominator of its rational part is zero. So $t - 2 \neq 0$, $t \neq 2$.

The function is defined at $x = 3$ so $x = 3$ is in the domain of the function.

For $x > 3$, the function presents no problems since it is the line $3x + 1$.

Solution: Domain of f is all real numbers except $x = 2$ (or $(-\infty, 2) \cup (2, \infty)$).

b)

A candidate for vertical asymptote $x = 2$ (the zero of the denominator of the rational part of f). Need to check that,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 + 1}{x - 2} = -\infty,$$

since $2^2 + 1 = 5 \neq 0$, and for $x = 1$, $\frac{1^2+1}{1-2} = -2 < 0$.

Solution: $x = 2$ is a vertical asymptote.

(You could also check,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2 + 1}{x - 2} = +\infty,$$

since $2^2 + 1 = 5 \neq 0$, and for $x = 2.5$, $\frac{(2.5)^2+1}{2.5-2} > 0$.)

c)

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 + 1}{x - 2} = \frac{3^2 + 1}{3 - 2} = 10,$$

and

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 3x + 1 = 3(3) + 1 = 10.$$

Therefore, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = 10$.

Solution: $\lim_{x \to 3} f(x) = 10$.

d)

f is continuous at $x = 3$ if $\lim_{x \to 3} f(x) = f(3)$. By c) $\lim_{x \to 3} f(x) = 10$ and

$$f(3) = \frac{3^2 + 1}{3 - 2} = 10.$$

So, $\lim_{x \to 3} f(x) = f(3)$.

Solution: f is continuous at $x = 3$.