Math 124 C,D (Hoffman)
October 22, 2002
Midterm #1 (70 points)

Name ______________

TA: ______________

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Section: ______________

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Instructions:

- Your exam contains 5 problems plus one extra credit problem. The entire exam is worth 70 points. The point value of each problem is clearly marked.
- Your exam should contain 7 pages; please make sure you have a complete exam.
- Box in your final answer when appropriate.
- When appropriate, carry out calculations to at least four decimal places.
- You have 80 minutes for this midterm. Unless stated otherwise, you **MUST** show work for credit. No credit for answers only. If in doubt, ask for clarification.
- Make sure to do your own work on the exam.

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Problem #1 (15 pts) ______

Problem #2 (15 pts) ______

Problem #3 (20 pts) ______

Problem #4 (15 pts) ______

Problem #5 (5 pts) ______

Bonus Problem ______

TOTAL (70 pts) ______

1. (20 pts) Find the derivatives of the following functions

   (a) (5 pts) $h(x) = 7(x^2 + 3x)$

$$h'(x) = 7(x^2 + 3x)' = 7(2x + 3) = 14x + 21.$$

   (b) (10 pts) $g(x) = \frac{xe^x}{x^2-3} - \cos(4x)$
   By the difference rule

$$g'(x) = \left(\frac{xe^x}{x^2 - 3}\right)' - (\cos(4x))'.$$

   By the chain rule

$$(\cos(4x))' = -4\sin(x).$$

   By the quotient rule

$$\begin{aligned}
\left(\frac{xe^x}{x^2 - 3}\right)' &= \frac{(x^2 - 3)(xe^x)' - (xe^x)(2x)}{(x^2 - 3)^2} \\
&= \frac{(x^2 - 3)(xe^x + e^x) - (2x^2 e^x)}{(x^2 - 3)^2} \\
&= \frac{e^x(x^3 - x^2 - 3x - 3)}{(x^2 - 3)^2}
\end{aligned}$$

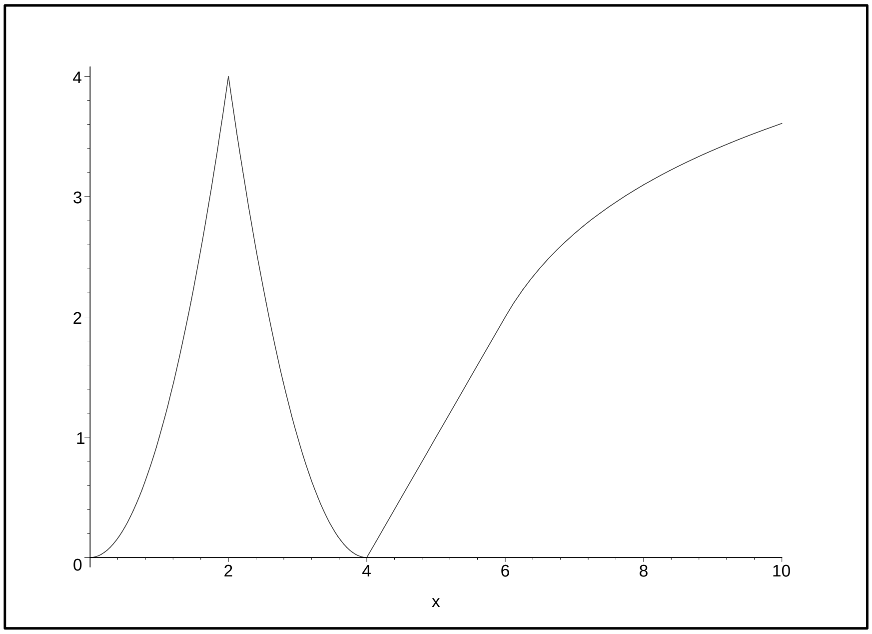
   Thus

$$g'(x) = \frac{e^x(x^3 - x^2 - 3x - 3)}{(x^2 - 3)^2} + 4\sin(x).$$

   (c) (5 pts) $h(x) = \ln(\cos(2x + 3))$
   By the chain rule $(\cos(2x + 3))' = -2\sin(2x + 3)$. Also by the chain rule

$$\ln(\cos(2x + 3)) = \frac{1}{\cos(2x + 3)}(\cos(2x + 3))' = \frac{-2\sin(2x + 3)}{\cos(2x + 3)}.$$

2. The graph of $y = f(x)$ is given below.




(a) (3 pts) Estimate $f'(1)$.

This is the slope of the tangent line at $(1, f(1))$. From the graph this appears to be about $\boxed{2}$.

(b) (3 pts) Which values of $x$, if any, is $f'(x)$ not defined.

The derivative is not defined if the graph is discontinuous, has sharp corners or has a vertical tangent line. There are no discontinuities or vertical tangent lines, but sharp corners at $\boxed{x = 2 \text{ and } x = 4}$

For each of the following pair determine whether the first number is greater than, less than, or equal to the second number. You do not need to show any work, just circle the appropriate symbol.

(c) (3 pts) $f''(5)$ $\quad >$ $\quad \boxed{=}$ $\quad <$ $\quad 0$

The graph is a line from $x = 4$ to $x = 6$. Thus the derivative is constant in this region and the second derivative is 0.

(d) (3 pts) $f(8.01)$ $\quad >$ $\quad =$ $\quad \boxed{<}$ $\quad f(8) + (.01)f'(8)$

The linear approximation to $f$ at $x = 8$ is

$$L(x) = f(8) + (x - 8)f'(8).$$

Thus the right hand side is $L(8.01)$. As the graph is concave downward at $x = 8$ the tangent line lies above the graph. Thus

$$f(8.01) < L(8.01) = f(8) + (.01)f'(8).$$

(e) (3 pts) $f'(5)$ $\quad \boxed{>}$ $\quad =$ $\quad <$ $\quad f'(8)$

The slope of the tangent line at $(5, f(5))$ is greater than the slope of the tangent line at $(8, f(8))$. Thus $f'(5) > f'(8)$

3. (15 pts) A search light in a prison rotates counterclockwise at a rate of 3 revolutions per second. The light shines on a long straight wall that is 40 feet from the search light. How fast (in ft/sec) is the light beam moving across the wall when the beam is hitting the wall at a spot which is 80 feet from the light?

We let $\theta(t)$ represent the angle between the line from the light to the point on the wall closest to the light and the line from the light to the point where the light beam hits the wall at time $t$. (We measure counterclockwise.)

We let $x(t)$ represent the location of the light beam at time $t$. ($x(t) = 0$ corresponds to the light pointing at the point on the wall which is closest to the light. This means $\theta(t) = 0$ or $2\pi$, $4\pi$,...)

We are looking for the speed of the light which is $|x'(t)|$ when the beam is hitting the wall 80 feet from the light.

We get the equation

$$x(t) = 40\tan(\theta(t)).$$

Differentiating both sides we get

$$x'(t) = \frac{40}{(\cos(\theta(t)))^2}\theta'(t).$$

When the light is 80 feet from the wall we have that $(80)^2 = (40)^2 + (x(t))^2$ or $x(t) = \pm 40\sqrt{3}$. Also $\cos(\theta(t)) = \frac{40}{80} = \frac{1}{2}$. Since the light is rotating at a constant rate $\theta'(t)$ is a constant. As the light makes a full revolution every 3 seconds, $\theta$ increases by $6\pi$ every second and $\theta'(t) = 6\pi$.

Plugging this into the equation above we get

$$x'(t) = \frac{40}{(1/2)^2}(6\pi) = 960\pi \approx 3015 \text{ ft/sec}.$$

4. A function $y = f(x)$ is defined implicitly by the equation

$$x^2y - y^3 + \sin(2\pi(x+y)) = 0.$$

(a) (10 pts) Find $\frac{dy}{dx}$ at the point $(2, 2)$.

$$\begin{aligned} x^2y - y^3 + \sin(2\pi(x+y)) &= 0 \\ \left(2xy + \frac{dy}{dx}x^2\right) - 3y^2\frac{dy}{dx} + \cos(2\pi(x+y))\left(2\pi\left(1+\frac{dy}{dx}\right)\right) &= 0 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx}x^2 - 3y^2\frac{dy}{dx} + \cos(2\pi(x+y))\left(2\pi\frac{dy}{dx}\right) &= -2xy - \cos(2\pi(x+y))(2\pi) \\ \frac{dy}{dx}\left(x^2 - 3y^2 + \cos(2\pi(x+y))\,(2\pi)\right) &= -2xy - \cos(2\pi(x+y))(2\pi) \\ \frac{dy}{dx} &= \frac{-2xy - \cos(2\pi(x+y))(2\pi)}{x^2 - 3y^2 + \cos(2\pi(x+y))\,(2\pi)} \end{aligned}$$

When $x = y = 2$ we have

$$\frac{dy}{dx} = \frac{-8 - \cos(8\pi)(2\pi)}{4 - 12 + \cos(8\pi)\,(2\pi)} = \frac{-8-2\pi}{-8+2\pi}$$

(b) (5 pts) Estimate what $y$ is when $x = 1.9973$.

$$L(x) = f(a) + f'(a)(x-a) = 2 + \frac{-8-2\pi}{-8+2\pi}(1.9973 - 2) = 2 + \frac{-8-2\pi}{-8+2\pi}(.0027).$$

5. (5 pts) Evaluate

$$\lim_{x\to 2} \frac{(7x+3)^{1/3} - (17)^{1/3}}{x-2}.$$

Plugging answers into your calculator is not justification for your answer.

From the definition of the derivative if $f(x) = (7x+3)^{1/3}$ then

$$f'(2) = \lim_{x\to 2} \frac{(7x+3)^{1/3} - (17)^{1/3}}{x-2}.$$

Since $f'(x) = \frac{1}{3}(7x+3)^{-2/3}(7) = \frac{7}{3}(7x+3)^{-2/3}$ we have that

$$\lim_{x\to 2} \frac{(7x+3)^{1/3} - (17)^{1/3}}{x-2} = f'(2) = \frac{7}{3}(7(2)+3)^{-2/3} = \frac{7}{3}(17)^{-2/3}.$$

6. (Bonus 10 pts) Find a function $y = f(x)$ that satisfies the differential equation

$$y'' - 2y' + 3y = 24e^{3x}.$$

(Hint: Think about the homework problem where you solved the differential equation $y'' + y' - 2y = \sin(x)$.)

In the homework problem we guessed that the solution was of the form $y = A\sin(x) = B\cos(x)$ and then solved for $A$ and $B$. In this problem we guess that $y = Ae^{3x}$ and will solve for $A$.

With this guess we get $y = Ae^{3x}$ then $y' = 3Ae^{3x}$ and $y'' = 9Ae^{3x}$. Then

$$24e^{3t} = y'' - 2y' + 3y = 9Ae^{3x} - 2(3Ae^{3x}) + 3(Ae^{3x}) = 6Ae^{3x}.$$

Thus we get $A = 4$. Thus $y = 4e^{3x}$.