1 (13 points) Compute the derivatives of the following functions. No need to simplify your answers.

a) $f(t) = (t^2 + 5)\arctan(3t)$

Product Rule, then Chain Rule:

$$f'(t) = 2t\arctan(3t) + (t^2 + 5)\frac{1}{1 + (3t)^2}(3)$$

b) $g(x) = \ln(x - \sqrt{1 + x^2})$

Chain Rule (twice):

$$g'(x) = \frac{1}{x - \sqrt{1 + x^2}}\left(x - \sqrt{1 + x^2}\right)' = \frac{1}{x - \sqrt{1 + x^2}}\left(1 - \frac{1}{2\sqrt{1 + x^2}}(2x)\right)$$

c) $y = \left(\frac{1}{x}\right)^{\sec x}$

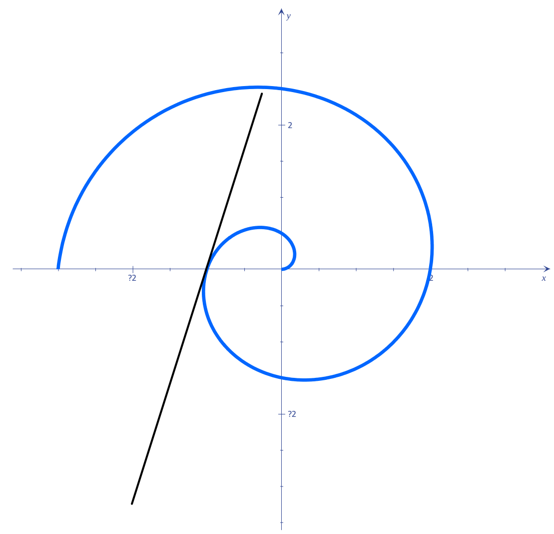
Since the variable appears both in the base and in the exponent, we need to use logarithmic differentiation:

$$\ln y = \sec x \ln\left(\frac{1}{x}\right)$$

$$\ln y = -\sec x \ln x$$

$$\frac{1}{y}y' = -[sec\, x \tan x \ln x + \sec x \frac{1}{x}]$$

$$\boxed{y' = -\left(\frac{1}{x}\right)^{\sec x}[sec\, x \tan x \ln x + \frac{\sec x}{x}]}$$



2 (12 points) Consider the spiral curve given by the following parametric equations, with $0 \le t \le 3$:

$$\begin{cases} x(t) = t\cos(\pi t) \\ y(t) = t\sin(\pi t) \end{cases}$$

a) Compute the horizontal and vertical velocities (as functions of t).

Computing the derivatives with respect to t. Use product rule (and chain):

$$\begin{cases} x'(t) = \cos(\pi t) - t\sin(\pi t)\,(\pi t)' \\ y'(t) = \sin(\pi t) + t\cos(\pi t)\,(\pi t)' \end{cases}$$

$$\boxed{\begin{cases} x'(t) = \cos(\pi t) - \pi t\sin(\pi t) \\ y'(t) = \sin(\pi t) + \pi t\cos(\pi t) \end{cases}}$$

b) Compute the equation of the tangent line to this curve at the point corresponding to $t = 1$.

The point on this spiral corresponding to t=1 is $\big(x(1), y(1)\big) = (1\cos(\pi), 1\sin(\pi)) = (-1, 0)$.

The slope of the tangent line at that point is given by: $\left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)} = \frac{\sin(\pi)+\pi\cos(\pi)}{\cos(\pi)-\pi\sin(\pi)} = \frac{0-\pi}{-1-0} = \pi$

The line through (-1,0) with slope π has equation:

$$\boxed{y = \pi(x+1) = \pi x + \pi}$$

c) Find **all** times t, $0 \le t \le 3$, at which the curve crosses the y-axis.

A curve crosses the y-axis when x=0.

Here, $x(t) = t\cos(\pi t) = 0$ is true when either $\boxed{t = 0}$, or $\cos(\pi t) = 0$

Solving $\cos(\pi t) = 0$ we get:

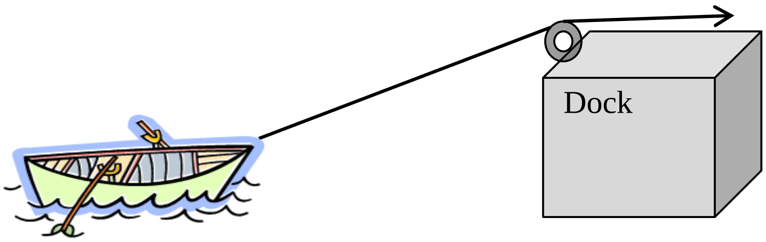
$\pi t = \frac{\pi}{2} \pm k\pi$, i.e. (dividing by π)

$t = \frac{1}{2} \pm k$, where k is any integer.

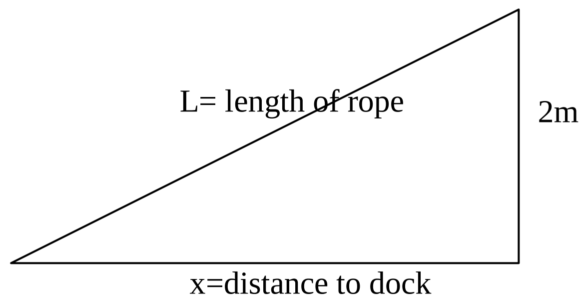
In the given interval $0 \le t \le 3$, we have $\boxed{t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}}$

(if you look at the picture: t=0 is the point at the origin, and the other 3 points are the y-intercepts along the curve)

3 (10 points) A boat is pulled towards a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 2 m higher than the bow of the boat. If the rope is pulled in at a rate of 1.5 m/sec, how fast is the boat approaching the dock when it is 6 m from the dock?



We have a right triangle as pictured:



We know: $\frac{dL}{dt} = -1.5\ m/s$.

We want: $\frac{dx}{dt}$=?

By the Pythagorean Theorem:

$$L^2 = x^2 + 2^2$$

Differentiating with respect to time t:

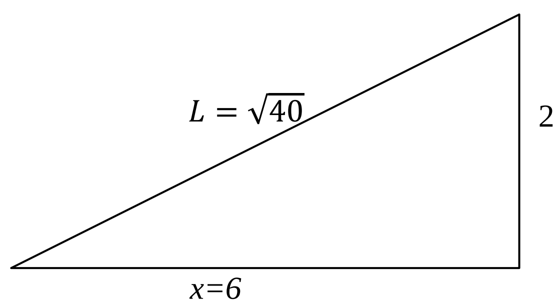
$$2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 0$$

Simplifying:

$$\frac{dx}{dt} = \frac{L}{x}\frac{dL}{dt}$$

When the boat is 6 m from the dock, x=6 so

$L^2 = 6^2 + 2^2$ ➔ $L = \sqrt{40} = 2\sqrt{10}$



So:

$$\left.\frac{dx}{dt}\right|_{x=6} = \frac{\sqrt{40}\ m}{6\ m}(-1.5)m/s = \boxed{\frac{-\sqrt{40}}{4} = \frac{-\sqrt{10}}{2}\ m/s}$$

4 (7 points) Let $f(x) = (1+x)^n$, where n is a constant.

a) Compute the linearization of $f(x)$ at $x = 0$. Your answer will depend on n.

$$L(x) = f'(0)(x-0) + f(0)$$

$$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n(1+0)^{n-1} = n$$
$$f(0) = (1+0)^n = 1$$

So $\boxed{L(x) = nx + 1}$

b) Use the linearization you found in part (a) to approximate $\sqrt[3]{1.009}$. Show all work, not just the final answer.

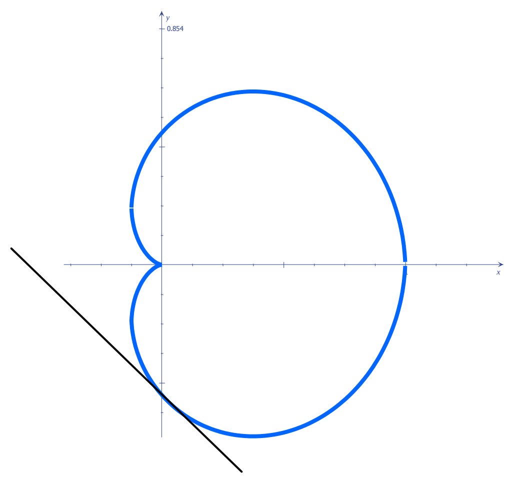
$\sqrt[3]{1.009} = (1+0.009)^{1/3}$, so we can use $x = 0.009,\ k = \frac{1}{3}$ in the above linearization:

$$\sqrt[3]{1.009} \approx \frac{1}{3}(0.009) + 1 = \boxed{1.003}$$

5 (8 points) Consider the cardioid curve given by the equation:

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

Find the equation of the tangent line to this curve at the point $\left(0, -\frac{1}{2}\right)$.



Implicitly differentiating the equation (with respect to x):

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)\left(4x + 4y\frac{dy}{dx} - 1\right)$$

Substituting x=0 and y= - 1/2, we can compute the derivative $\frac{dy}{dx}$ at the point $\left(0, \frac{-1}{2}\right)$:

$$0 + 2\frac{-1}{2}\frac{dy}{dx} = 2\left(0 + 2\frac{1}{4} - 0\right)\left(0 + 4\frac{-1}{2}\frac{dy}{dx} - 1\right),$$

which we solve to get $\left.\frac{dy}{dx}\right|_{(0,\frac{-1}{2})} = -1.$

Hence, the equation of tangent line at $\left(0, \frac{-1}{2}\right)$ is: $\boxed{y = -x - \frac{1}{2}}$