Worksheet #2 - Solutions

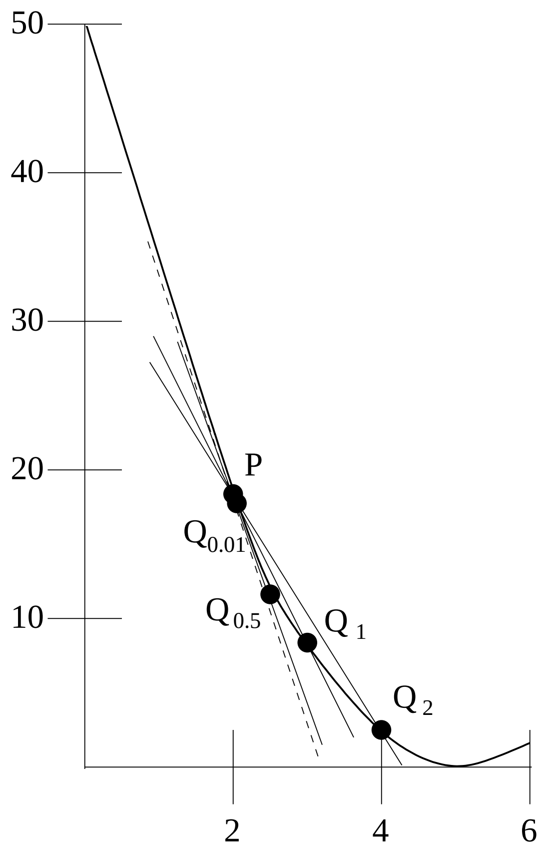
Average and Instantaneous Velocity — Math 124

Introduction

In this worksheet, we introduce what are called the *average and instantaneous velocity* in the context of a specific physical problem: A golf ball is hit toward the cup from a distance of 50 feet. Assume the distance from the ball to the cup at time t seconds is given by the function

$$d(t) = 50 - 20t + 2t^2.$$

Here is the graph of $y = d(t)$.



1. Does the ball reach the cup? If so, when? (Answer this question two ways: by using algebra, and by reading the graph.)

   The ball will reach the cup if the distance is ever zero. From the graph, it appears that the distance is zero when $t = 5$; to be precise, we check this mathematically. Using the **quadratic formula**, we see that $50 - 20t + 2t^2 = 0$ when $t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(50)(2)}}{2(2)}$, that is, when $t = 5$.

   Thus, the ball reaches the cup at time $t = 5$.

2. (a) Plot and label these points on the graph:

$$\begin{aligned} P &= (2, d(2)) \\ Q_2 &= (4, d(4)) \\ Q_1 &= (3, d(3)) \\ Q_{0.5} &= (2.5, d(2.5)) \\ Q_{0.01} &= (2.01, d(2.01)) \end{aligned}$$

   See graph above.

(b) Sketch the line through P and Q_2
Sketch the line through P and Q_1
Sketch the line through P and $Q_{0.5}$
Sketch the line through P and $Q_{0.01}$

See graph on page 1. The solid lines plotted there are the first three lines listed here; the last one listed here is omitted for visual clarity.

(c) Compute the slopes of the lines in (b).
Slope of the line through P and Q_2: $m = \frac{d(4)-d(2)}{4-2} = \frac{2-18}{2} = \boxed{-8}$.
Slope of the line through P and Q_1: $m = \frac{d(3)-d(2)}{3-2} = \frac{8-18}{1} = \boxed{-10}$.
Slope of the line through P and $Q_{0.5}$: $m = \frac{d(2.5)-d(2)}{2.5-2} = \frac{12.5-18}{0.5} = \boxed{-11}$.
Slope of the line through P and $Q_{0.01}$: $m = \frac{d(2.01)-d(2)}{2.01-2} = \frac{17.8802-18}{0.01} = \boxed{-11.98}$.

(d) We define the average velocity as follows: average velocity$=\frac{\text{distance traveled}}{\text{time elapsed}}$.
Explain why the slope of the line $\overline{PQ_2}$= the average velocity from $t = 2$ seconds to $t = 4$ seconds.
The distance travelled is equal to (final position)-(initial position). Therefore, the distance travelled from $t = 2$ to $t = 4$ is $d(4) - d(2)$. Also, the time elapsed is $4 - 2$. Consequently, the average velocity is

$$\boxed{average\ velocity = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{d(4)-d(2)}{4-2} = slope\ of\ line\ \overline{PQ_2}}$$

(e) Find the average velocity over the following time intervals:
From part (d), we know that the average velocities are the same as the slopes, so we can copy our answers from part (c), adding units:
$t = 2$ seconds to $t = 4$ seconds: $\boxed{-8}$ feet/second
$t = 2$ seconds to $t = 3$ seconds: $\boxed{-10}$ feet/second
$t = 2$ seconds to $t = 2.5$ seconds: $\boxed{-11}$ feet/second
$t = 2$ seconds to $t = 2.01$ seconds: $\boxed{-11.98}$ feet/second

(f) The average velocities in (e) approach a number as the time interval gets smaller and smaller. Guess this number.
The answers in part (e) appear to approach the value $\boxed{-12}$ feet/second as the time intervals get shorter.

3. Let h be a small constant positive number and define $Q_h = (2+h, d(2+h))$. Compute the slope of the secant line connecting P and Q_h by simplifying:

$$slope = \frac{(\text{y coordinate}\, Q_h) - (\text{y coordinate}\, P)}{(\text{t coordinate}\, Q_h) - (\text{t coordinate}\, P)}$$

so there is no h in the denominator. This slope = the average velocity on the time interval $t = 2$ seconds to $t = 2 + h$ seconds.

First, let's simplify the expression for the y-coordinate of Q_h:

$$\begin{aligned} d(2+h) &= 50 - 20(2+h) + 2(2+h)^2 \\ &= 50 - 40 - 20h + 2(4 + 4h + h^2) \\ &= 18 - 12h + 2h^2 \end{aligned}$$

If we substitute the formulas for the coordinates of Q_h and P into the formula above, we obtain

$$slope = \frac{d(2+h) - d(2)}{(2+h) - 2} = \frac{(18 - 12h + 2h^2) - (18)}{h} = \frac{2h^2 - 12h}{h} = 2h - 12$$

That is, $\boxed{\text{slope} = 2h - 12}$.

4. What number do you get when you plug $h = 0$ into the simplified expression in problem 3 above? This is called the instantaneous velocity at $t = 2$ seconds.

   Plugging in $h = 0$ to the expression computed above, we get

$$\boxed{slope = 2(0) - 12 = 12}.$$

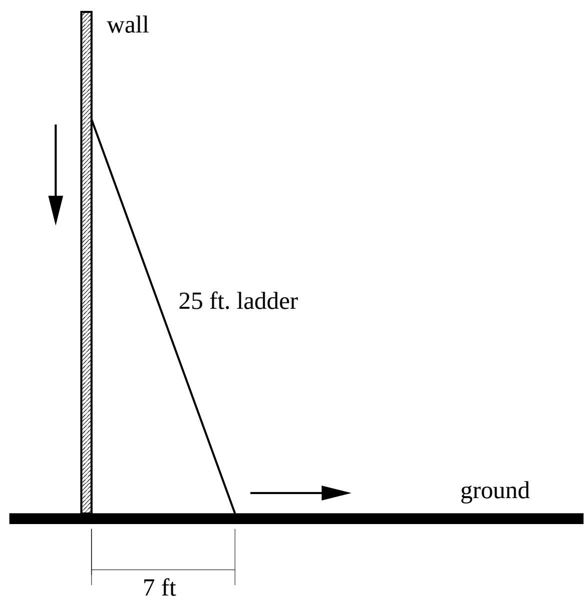
5. Draw a line through P with slope equal to the number computed in 5. How would you describe this line relative to the graph?

   The dashed line drawn in the figure is the one described here.

   $\boxed{\text{The line appears to be tangent to the graph at } P.}$

If your group has finished, start working on this problem from homework #2:

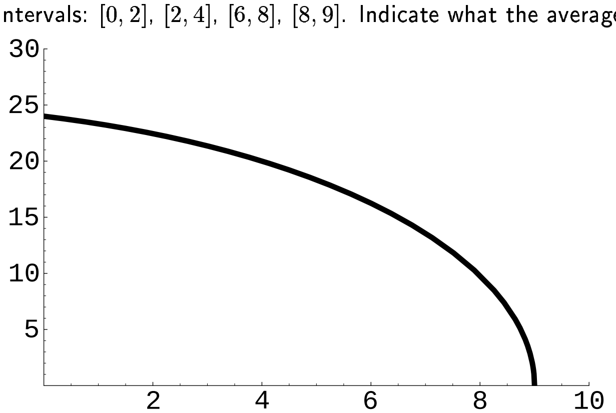
A ladder 25 feet long is leaning against the wall of a building. Initially, the foot of the ladder is 7 feet from the wall. The foot of the ladder begins to slide at a rate of 2 ft/sec, causing the top of the ladder to slide down the wall. The location of the foot of the ladder at time t seconds is $(7+2t, 0)$.



(a) The location of the top of the ladder will be $(0, y(t))$, for some function $y(t)$. Find the formula for $y(t)$. What is the domain of t values?

The function $y(t)$ represents height of the top of the ladder (where it touches the wall). Observe from the picture that the ladder, the ground and the wall form a right triangle. The ladder is 25 feet long, and the bottom of the ladder is $7+2t$ feet from the wall. Therefore, by the Pythagorean Theorem, the top of the ladder is at the height $\sqrt{25^2-(7+2t)^2} = \sqrt{625-(49+28t+4t^2)} = \sqrt{576-28t-4t^2}$. That is, $\boxed{y(t)=\sqrt{576-28t-4t^2}}$. This makes sense until the moment when the top of the ladder reaches the ground (so that the ladder has completely fallen on the floor). That occurs when the bottom of the ladder is 25 feet from the wall. Solving the equation $7+2t=25$ gives us $t=9$. Consequently, $\boxed{\text{the domain is } 0 \le t \le 9}$.

(b) The graph of the function $y(t)$ is given below. Compute the average velocity of the top of the ladder on these time intervals: $[0,2]$, $[2,4]$, $[6,8]$, $[8,9]$. Indicate what the average velocity is telling you in terms of the picture.



$[0,2]$: $v_{avg} = \frac{y(2)-y(0)}{2-0} = \frac{22.45-24}{2} = \boxed{-0.775}$

$[2,4]$: $v_{avg} = \frac{y(4)-y(2)}{4-2} = \frac{20-22.45}{2} = \boxed{-1.225}$

$[6,8]$: $v_{avg} = \frac{y(8)-y(6)}{8-6} = \frac{9.8-16.25}{2} = \boxed{-3.225}$

$[8,9]$: $v_{avg} = \frac{y(9)-y(8)}{9-8} = \frac{0-9.8}{1} = \boxed{-9.8}$

The units of all answers are $\frac{\text{feet}}{\text{second}}$.

Each average velocity also tells us the slope of a secant line intersecting the graph at the times corresponding to the endpoints of the interval.

(c) The foot of the ladder is moving at a constant rate; how about the top of the ladder?

The average velocities are increasing, so $\boxed{\text{the top of the ladder falls faster and faster.}}$