

Worksheet #3 - Solutions

Introduction

This worksheet discusses a method of computing limits for some special functions.

1. Use the graph of the function f at right to answer the following questions. Assume this is the entire graph of f .

(a) What is the domain of f ?

The domain is $\boxed{-2 \leq x < 0 \text{ and } 0 < x \leq 6}$.

(b) Find $\lim_{x \rightarrow 3} f(x)$.

From the graph, we see that

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = 2,$$

and therefore, $\boxed{\lim_{x \rightarrow 3} f(x) = 3}$.

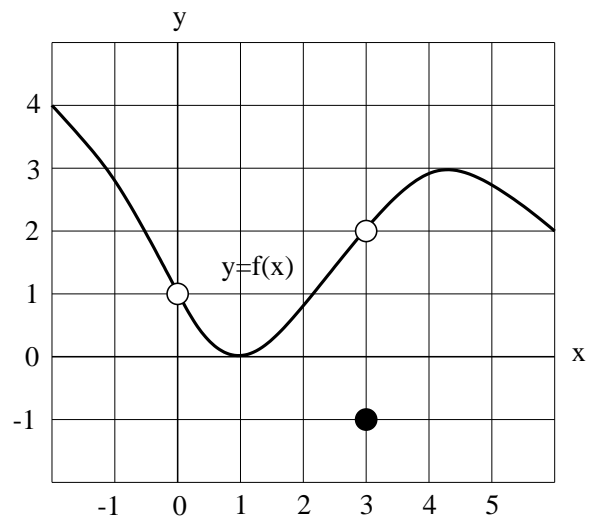
(c) Find $\lim_{x \rightarrow 0} f(x)$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 1.$$

Therefore, $\boxed{\lim_{x \rightarrow 0} f(x) = 1}$.

(d) What is $f(3)$?

$$\boxed{f(3) = -1}$$



DEFINITION: A function g is called **continuous at** a if two things are true:

- (i) a is a point in the domain of g and (ii) $\lim_{x \rightarrow a} g(x) = g(a)$.

A function is just called **continuous** if it is continuous at every point in its domain.

2. Answer the following questions for the function f whose graph is above. Explain your answers.

(a) Is f continuous at 3? $\boxed{\text{No.}}$ Since $\lim_{x \rightarrow 3} f(x) = 3$ and $f(3) = -1$, we see that $f(3) \neq \lim_{x \rightarrow 3} f(x)$.

(b) Is f continuous at 0? $\boxed{\text{No.}}$ Since 0 is not in the domain of f .

(c) Is f continuous at 1? $\boxed{\text{Yes.}}$ We see from the graph that $f(1) = 0$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = 0$. Therefore, $f(1) = \lim_{x \rightarrow 1} f(x)$.

(d) Is f continuous? $\boxed{\text{No.}}$ Since f fails to be continuous at a point in its domain (i.e. at 3), f is not continuous.

Not every function is continuous, but many are. It's often important to know ahead of time that a function we want to work with is continuous, because that makes limit calculations easy. The following kinds of functions are *always* continuous:

- constant functions
- linear functions
- polynomials
- trigonometric functions
- exponential functions

There are many other kinds of continuous functions; some will be explored later in the course. You can also combine continuous functions to make new continuous functions. For example, you can add, multiply or compose continuous functions and be guaranteed to get a continuous function. Division, on the other hand, sometimes creates complications, as we'll see below.

3. Find $\lim_{x \rightarrow 2} 3x^2 - 2x + 4$. Explain your solution.

The function $3x^2 - 2x + 4$ is a polynomial. Therefore, by the previous remarks, it is continuous. Consequently, we can find the limit by evaluating the function at the desired point:

$$\lim_{x \rightarrow 2} 3x^2 - 2x + 4 = 3(2)^2 - 2(2) + 4 = 3(4) - 4 + 4 = \boxed{12}.$$

4. Can you find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ by plugging in $x = 1$? Explain.

No. You can't plug in $x = 1$ because it's not in the domain of the function $\frac{x^2 - 1}{x - 1}$.

(It fails to be in the domain because the denominator would be zero.)

Sometimes we can find a limit of a function where it isn't continuous or defined by first simplifying the function, so that it resembles a continuous function.

5. Simplify the function in problem number 4, and find the limit of the resulting expression as $x \rightarrow 1$. Why can you say that this is the same as the limit you were asked to find in problem number 4?

Simplifying yields

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1, \quad \text{if } x \neq 1.$$

The limit as x approaches 1 depends only on the values of $f(x)$ for x close to 1, not including $x = 1$. Since the two expressions $\frac{x^2 - 1}{x - 1}$ and $x + 1$ are equal for all values of $x \neq 1$, these two functions have the same limit as x approaches 1. That is,

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} x + 1.$$

The expression, $x + 1$, is continuous at every real number because it is linear and its domain is the set of all real numbers. Therefore we can find the limit by evaluating the function at the given point, and we conclude that

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} x + 1 = 1 + 1 = \boxed{2}.$$

6. Let $e(x) = 125x^2 - 6.25x^3$.

- (a) Simplify the expression $\frac{e(5+h)-e(5)}{(5+h)-(5)}$ as much as possible; there should not be an h in the denominator of any fraction in the simplified expression.

$$\begin{aligned} \frac{e(5+h) - e(5)}{(5+h) - (5)} &= \frac{125(5+h)^2 - 6.25(5+h)^3 - 125(5^2) + 6.25(5^3)}{h} \\ &= \frac{(125(5^2 + 2(5)h + h^2) - 6.25(5^3 + 3(5)h^2 + 3(5^2)h + h^3) - 125(5^2) + 6.25(5^3))}{h} \\ &= \frac{(1250h + 125h^2 - 93.75h^2 - 468.75h - 6.25h^3)}{h} \\ &= 1250 + 125h - 93.75h - 468.75 - 6.25h^2 \\ &= 781.25 + 31.25h - 6.25h^2 \end{aligned}$$

- (b) Is $\frac{e(5+h)-e(5)}{(5+h)-(5)}$ continuous at $h = 0$? Why?

Is the simplified expression in part (a) continuous at $h = 0$? Why?

The expression $\frac{e(5+h)-e(5)}{(5+h)-(5)}$ is not continuous at $h = 0$ since 0 is not in its domain (the denominator would be zero). On the other hand, the simplified expression is a polynomial, which has all real numbers in its domain and is continuous at every real number.

- (c) Calculate $\lim_{h \rightarrow 0} \frac{e(5+h) - e(5)}{(5+h) - (5)}$ using the approach outlined in problem 5.

The expression $\frac{e(5+h)-e(5)}{(5+h)-(5)}$ has the same limit as $h \rightarrow 0$ as does the simplified expression $781.25 + 31.25h - 6.25h^2$ since they are equal for $h \neq 0$. Also, since the polynomial is continuous, we can evaluate the limit as $h \rightarrow 0$ by plugging in $h = 0$. Therefore,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{e(5+h) - e(5)}{(5+h) - (5)} &= \lim_{h \rightarrow 0} 781.25 + 31.25h - 6.25h^2 \\ &= 781.25 + 31.25(0) - 6.25(0)^2 \\ &= 781.25 \end{aligned}$$

7. Let $e(x) = 125x^2 - 6.25x^3$, as in problem 6. Let x be an unknown constant.

- (a) Simplify the expression $\frac{e(x+h)-e(x)}{(x+h)-(x)}$ as much as possible; there should not be an h in the denominator of any fraction in the simplified expression.

$$\begin{aligned} \frac{e(x+h) - e(x)}{(x+h) - (x)} &= \frac{125(x+h)^2 - 6.25(x+h)^3 - 125(x^2) + 6.25(x^3)}{h} \\ &= \frac{(125(x^2 + 2xh + h^2) - 6.25(x^3 + 3xh^2 + 3x^2h + h^3) - 125(x^2) + 6.25(x^3))}{h} \\ &= \frac{(250xh + 125h^2 - 18.75xh^2 - 18.75x^2h - 6.25h^3)}{h} \\ &= 250x + 125h - 18.75xh - 18.75x^2 - 6.25h^2 \end{aligned}$$

(b) Is $\frac{e(x+h)-e(x)}{(x+h)-(x)}$ continuous at $h = 0$? Why?

Is the simplified expression in part (a) continuous at $h = 0$? Why?

The expression $\frac{e(x+h)-e(x)}{(x+h)-(x)}$ is not continuous at $h = 0$ since 0 is not in its domain (the denominator would be zero). On the other hand, the simplified expression is a polynomial, which has all real numbers in its domain and is continuous at every real number.

(c) Calculate $\lim_{h \rightarrow 0} \frac{e(x+h) - e(x)}{(x+h) - (x)}$ using the approach outlined in problem 5.

The expression $\frac{e(x+h)-e(x)}{(x+h)-(x)}$ has the same limit as $h \rightarrow 0$ as does the simplified expression $250x + 125h - 18.75xh - 18.75x^2 - 6.25h^2$ since they are equal for $h \neq 0$. Also, since the polynomial is continuous, we can evaluate the limit as $h \rightarrow 0$ by plugging in $h = 0$. Therefore,

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{e(x+h) - e(x)}{(x+h) - (x)} &= \lim_{h \rightarrow 0} 250x + 125h - 18.75xh - 18.75x^2 - 6.25h^2 \\ &= 250x + 125(0) - 18.75x(0) - 18.75x^2 - 6.25(0)^2 \\ &= 250x - 18.75x^2\end{aligned}$$

If your group has finished, start working on this problem from homework #3:

The location of an object on the x -axis at time t seconds is given by $x(t) = a + mt$, where a, m are constants.

1. Where is the object located at time $t = 1$?

Observe that $x(1) = a + m(1) = a + m$. Therefore, at time $t = 1$, the object is at the point $\boxed{x = a + m}$.

2. What is the average velocity of the object between 1 and 2 seconds?

Recall that average velocity is given by $\frac{\text{distance travelled}}{\text{time elapsed}}$. We need to compute the distance travelled; this is just the difference given by (final position - initial position). We have $x(2) = a + m(2) = a + 2m$. Therefore, the distance travelled is

$$x(2) - x(1) = (a + 2m) - (a + m) = m.$$

The time elapsed is $2 - 1 = 1$ second, so the average velocity is

$$\frac{x(2) - x(1)}{2 - 1} = \frac{m}{1} = \boxed{m}.$$

3. What is the instantaneous velocity of the object at time $t = 1$?

Let's first find the average velocity from time $t = 1$ to time $t = 1 + h$, where h is a small increment of time:

$$\begin{aligned}\text{average velocity} &= \frac{x(1+h) - x(1)}{(1+h) - (1)} \\ &= \frac{(a + m(1+h)) - (a + m(1))}{h} \\ &= \frac{mh}{h} \\ &= m.\end{aligned}$$

This is a constant that does not depend on h . Therefore, the instantaneous velocity at time $t = 1$ is

$$\lim_{h \rightarrow 0} (\text{average velocity from } t = 1 \text{ to } t = 1 + h) = \lim_{h \rightarrow 0} m = \boxed{m}$$

4. What is the instantaneous velocity of the object at time t ? We duplicate the steps in problem 3, this time leaving t unknown. The average velocity from time t to time $t + h$ is

$$\frac{x(t+h) - x(t)}{(t+h) - (t)} = \frac{(a + m(t+h)) - (a + mt)}{h} = \frac{mh}{h} = m,$$

which is again a constant that does not depend on h . Consequently, the instantaneous velocity at time t is

$$\lim_{h \rightarrow 0} m = \boxed{m}.$$