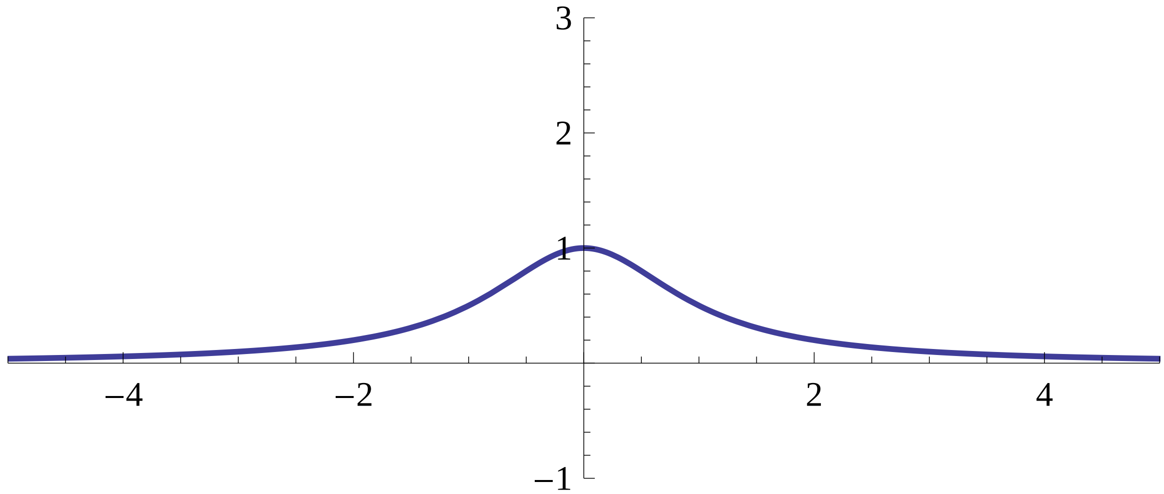
The Derivative Function Math 124

Introduction

This worksheet will work with the function $y = f(x) = \dfrac{1}{x^2+1}$ whose graph is given below:



Recall that the derivative of $f(x)$ at $x = a$, denoted by $f'(a)$, is the instantaneous rate of change of $f(x)$ at $x = a$, which is the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

1. Looking only at the graph of $y = f(x)$ above, answer these questions about $f'(a)$; you should be able to answer these questions without doing any calculuations:

   (a) For which a is $f'(a)$ positive?

   (b) For which a is $f'(a)$ negative?

   (c) For which a is $f'(a)$ zero?

   (d) What is $\lim_{a\to\infty} f'(a)$?

   (e) What is $\lim_{a\to-\infty} f'(a)$?

   (f) What is $\lim_{a\to 0} f'(a)$?

   (g) If you consider the slopes of all of the possible tangent lines to the graph of $y = f(x)$, is there a largest slope? If so, approximately where is this tangent line on the graph of $y = f(x)$? If not, why not.

   (h) If you consider the slopes of all of the possible tangent lines to the graph of $y = f(x)$, is there a smallest slope? If so, approximately where is this tangent line on the graph of $y = f(x)$? If not, why not.

2. Let h be a small real number, fix the point $P = (a, f(a))$ on the graph of $y = f(x)$ and the nearby point $Q = (a+h, f(a+h))$.

   (a) Let $s(h)$ = slope of the secant line through P and Q. Find a formula for $s(h)$. Simplify your answer so that it is a single rational expression in h and simplify it as much as possible.

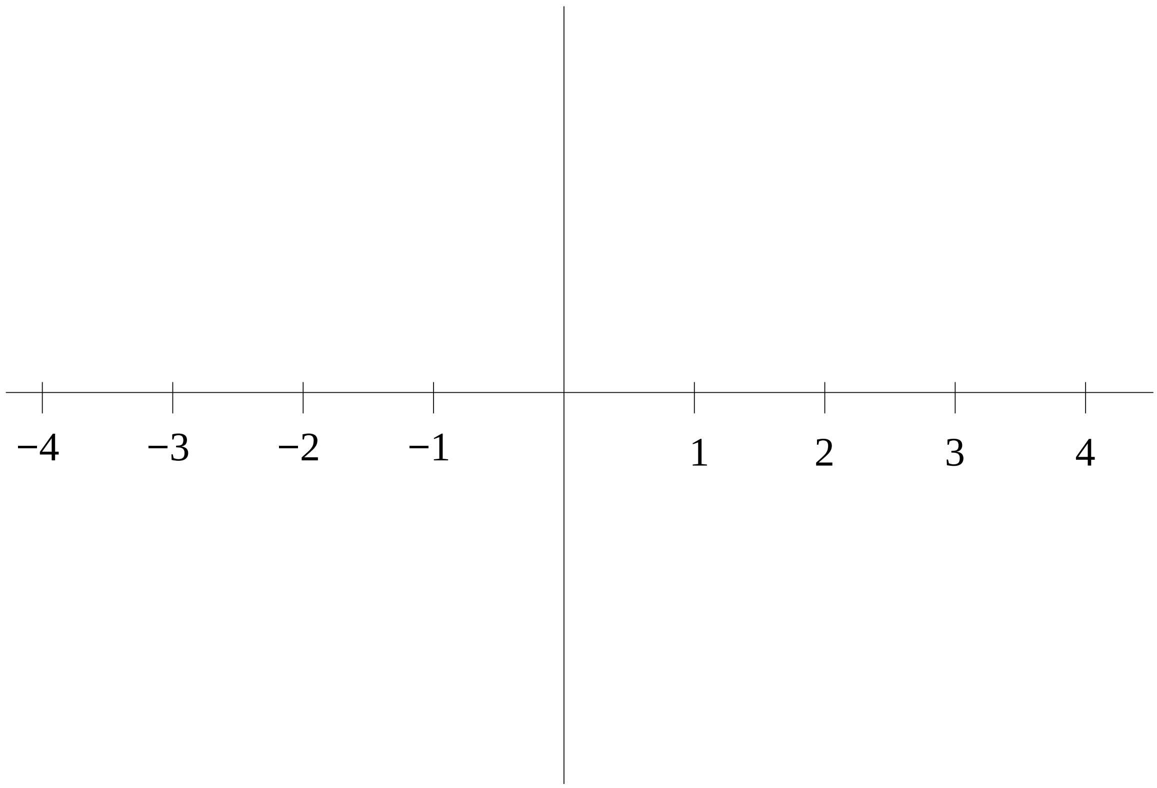
   (b) Calculate $f'(a) = \lim_{h \to 0} s(h)$.

   (c) Create a table of values:

a	$f'(a)$
-4	
-3	
-2	
-1	
-.9	
-.8	
-.7	
-.6	
-.5	
-.4	
-.3	
-.2	
-.1	
0	

a	$f'(a)$
.1	
.2	
.3	
.4	
.5	
.6	
.7	
.8	
.9	
1	
2	
3	
4	

(d) Plot the points $(a, f'(a))$ from your table in the previous step below and "connect the dots". Replacing "a" by "x" in 2(b), the sketch below is the graph of $y = f'(x) = \frac{-2x}{(1+x^2)^2}$; the *derivative function* $y = f'(x)$. Discuss how your answers in question 1 relate to this picture.



3. Starting with the function $y = g(x) = \dfrac{-2x}{(1+x^2)^2}$ and a real number $x = a$, follow the steps in 2(a),(b) to calculate the formula for $g'(a)$.

4. How can you use the work in 3. to exactly determine the locations on the graph of $y = f(x)$ where the tangent line has largest and smallest slope (i.e. the exact answers to 1(g),(h))?