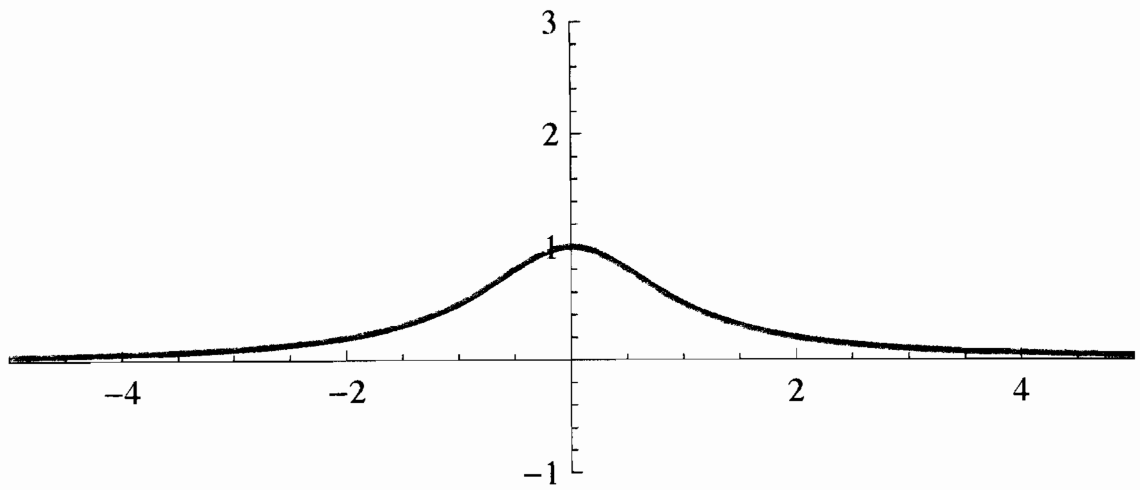
Worksheet (SOLUTION)

The Derivative Function

Math 124

Introduction

This worksheet will work with the function $y = f(x) = \frac{1}{x^2+1}$ whose graph is given below:



Recall that the derivative of $f(x)$ at $x = a$, denoted by $f'(a)$, is the instantaneous rate of change of $f(x)$ at $x = a$, which is the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

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1. Looking only at the graph of $y = f(x)$ above, answer these questions about $f'(a)$; you should be able to answer these questions without doing any calculuations:

   (a) For which a is $f'(a)$ positive? $-\infty < a < 0$

   (b) For which a is $f'(a)$ negative? $0 < a < \infty$

   (c) For which a is $f'(a)$ zero? 0

   (d) What is $\lim_{a\to\infty} f'(a)$? 0

   (e) What is $\lim_{a\to-\infty} f'(a)$? 0

   (f) What is $\lim_{a\to 0} f'(a)$? 0

   (g) If you consider the slopes of all of the possible tangent lines to the graph of $y = f(x)$, is there a largest slope? If so, approximately where is this tangent line on the graph of $y = f(x)$? If not, why not. YES; between -1 & 0

   (h) If you consider the slopes of all of the possible tangent lines to the graph of $y = f(x)$, is there a smallest slope? If so, approximately where is this tangent line on the graph of $y = f(x)$? If not, why not. YES; between 0 & 1

2. Let h be a small real number, fix the point $P = (a, f(a))$ on the graph of $y = f(x)$ and the nearby point $Q = (a+h, f(a+h))$.

(a) Let $s(h)$ = slope of the secant line through P and Q. Find a formula for $s(h)$. Simplify your answer so that it is a single rational expression in h and simplify it as much as possible.

$$s(h) = \frac{f(a+h) - f(a)}{a+h-a} = \frac{\frac{1}{(a+h)^2+1} - \frac{1}{a^2+1}}{h} = \frac{a^2+1-(a+h)^2-1}{((a+h)^2+1)(a^2+1)h}$$

$$= \frac{a^2+1-a^2-2ah-h^2-1}{((a+h)^2+1)(a^2+1)h} = \frac{-2ah-h^2}{((a+h)^2+1)(a^2+1)h}$$

$$= \boxed{\frac{-2a-h}{((a+h)^2+1)(a^2+1)}}$$

(b) Calculate $f'(a) = \lim_{h \to 0} s(h)$. $= \lim_{h \to 0} \frac{-2a-h}{((a+h)^2+1)(a^2+1)} = \frac{-2a}{(a^2+1)^2}$

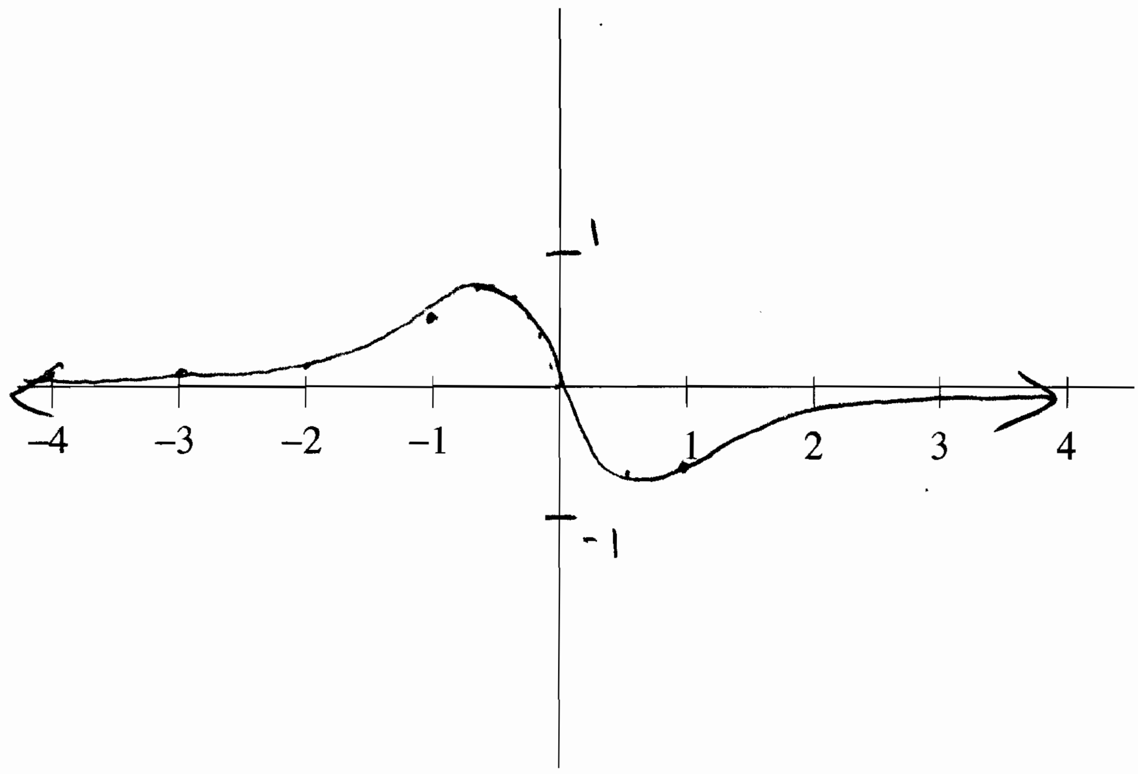
use continuity of $\square$ at $h=0$

(c) Create a table of values:

a	$f'(a)$
-4	0.0277
-3	0.06
-2	0.16
-1	0.5
-.9	0.5494
-.8	0.5949
-.7	0.6306
-.6	0.6488
-.5	0.64
-.4	0.5945
-.3	0.505
-.2	0.3698
-.1	0.1961
0	0

a	$f'(a)$
.1	-0.1961
.2	-0.3698
.3	-0.505
.4	-0.5945
.5	-0.64
.6	-0.6488
.7	-0.6306
.8	-0.5949
.9	-0.5494
1	-0.5
2	-0.16
3	-0.06
4	-0.0277

(d) Plot the points $(a, f'(a))$ from your table in the previous step below and "connect the dots". Replacing "a" by "x" in 2(b), the sketch below is the graph of $y = f'(x) = \frac{-2x}{(1+x^2)^2}$; the *derivative function* $y = f'(x)$. Discuss how your answers in question 1 relate to this picture.



Places where f'(x) graph above x-axis $\Leftrightarrow$ 1(a)
" " " " below " " $\Leftrightarrow$ 1(b)
" " " " crosses " " $\Leftrightarrow$ 1(c)

etc.

(g)(h) We can see a largest/smallest value occurs, though we don't have an exact place determined yet.

3. Starting with the function $y = g(x) = \dfrac{-2x}{(1+x^2)^2}$ and a real number $x = a$, follow the steps in 2(a),(b) to calculate the formula for $g'(a)$.

You shuld get

$$g'(a) = \frac{2(3a^2-1)(a^2+1)}{(a^2+1)^4}$$

4. How can you use the work in 3. to exactly determine the locations on the graph of $y = f(x)$ where the tangent line has largest and smallest slope (i.e. the exact answers to 1(g),(h))?

Looking at the graph in (2d), the max value of $g(x) = f'(x)$ occurs where the tan line to graph of $f'(x) = g(x)$ is horizontal. This happens when $g'(x) = 0$ and by 3, this happens when the factor $(3a^2-1) = 0$; i.e. $a = \pm\frac{1}{\sqrt{3}}$. This gives us the exact x-coordinates of the high & low points of the graph in (2d).