Worksheet-Week #6-Solutions

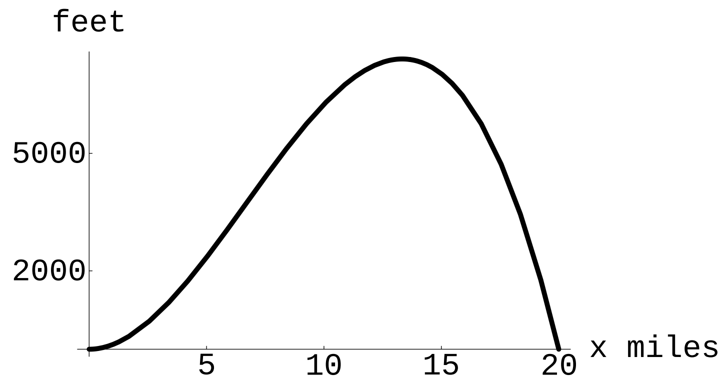
The Hiker and the Submarine — Math 124

Introduction

This worksheets introduces the problem of find maximum and minimum values of functions using techniques of calculus.

1. You are planning a 20 mile hike in the Cascades. The guidebook provides you with a chart, plotting elevation above the trailhead (in feet) as a function of the distance hiked (in miles). This plot is shown and is modeled by the function

$$e(x) = 125x^2 - 6.25x^3.$$



(a) Using the graph of $e(x)$, explain in words how the tangent lines to the graph relate to the difficulty of the hike.

The slope of a tangent line corresponds to the rate of increase in elevation per mile hiked. A greater rate of increase in elevation corresponds to a more strenuous (difficult) hike. A lesser rate of increase in elevation corresponds to an easier hike.

(b) What can you say about the slope of the tangent line at the point of highest elevation on the graph?

From the graph, we see that the tangent line at the point of highest elevation will be horizontal (i.e. it will have slope=0). This makes sense because at the highest point, the hiker has finished ascending but has not yet begun descending.

(c) Find the instantaneous rate of change of elevation as a function of $x =$ (distance hiked). Include units in your answer.

The rate of change of elevation $e(x)$ is the derivative of $e(x)$: $e'(x) = 250x - 18.75x^2$. The units of $e(x)$ are feet, and the units of x are miles, so the units of $e'(x)$ are $\frac{\text{feet}}{\text{mile}}$. That is to say, the rate of change of elevation with units is

$$\boxed{250x - 18.75x^2 \frac{\text{feet}}{\text{mile}}}.$$

(d) Determine how far the hiker will have traveled when he reaches the highest point on the trail.

From question (b), we know that the tangent line at the highest point has slope=0. From question (c), we know the formula for the slope of the tangent line is $e'(x) = 250x - 18.75x^2$. So we solve the equation $e'(x) = 0$ to find the x-coordinate of the point of highest elevation.

$$\begin{aligned} 0 &= e'(x) \\ &= 250x - 18.75x^2 \\ &= (250 - 18.75x)x \end{aligned}$$

This has solutions $x = 0$ and $x = \frac{250}{18.75} = \frac{40}{3} = 13.\overline{3}$. The first solution, $x = 0$ corresponds to the very beginning of the hike. Therefore, the solution $x = 13.\overline{3}$ must correspond to the point of highest elevation.

Consequently, we see that the hiker will have traveled $\boxed{\frac{40}{3} = 13.\overline{3} \text{ miles}}$ when he reaches the highest point on the trail.

(e) What is the elevation of the highest point on the trail? Give your answer accurate to the nearest whole foot.

The point of highest elevation occurs $13.\overline{3} = \frac{40}{3}$ miles into the hike. Since the function $e(x)$ tells us the elevation x miles into the hike, we can use it to compute the highest elevation along the trail:

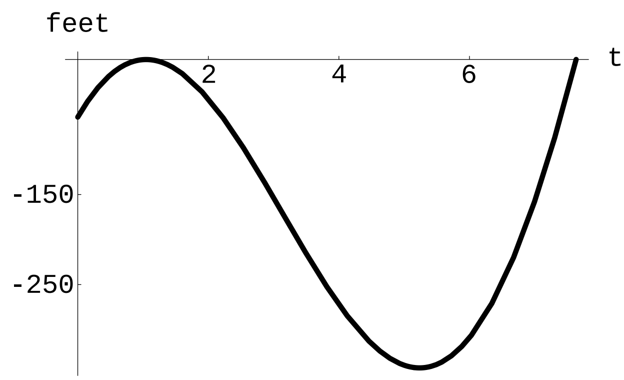
$$\begin{aligned} e\left(\frac{40}{3}\right) &= 125\left(\frac{40}{3}\right)^2 - 6.25\left(\frac{40}{3}\right)^3 \\ &= 125\left(\frac{1600}{9}\right) - 6.25\left(\frac{64000}{27}\right) \\ &= 125\left(\frac{1600}{9}\right) - \frac{25}{4}\left(\frac{64000}{27}\right) \\ &= \frac{200000}{9} - \frac{400000}{27} \\ &= \frac{600000}{27} - \frac{400000}{27} \\ &= \frac{200000}{27} \\ &= 7407.\overline{407} \\ &\approx 7407 \end{aligned}$$

Therefore, the highest elevation along the trail is approximately $\boxed{7407 \text{ feet}}$.

2. A submarine executes a diving drill. Starting from 64 feet below sea level, the submarine surfaces, then dives several hundred feet before resurfacing. The altitude (height above sea level) of the submarine in feet is given by the function

$$a(t) = 500\cos\left(\frac{t}{2}\right) + 125t - 564,$$

where t represents the number of minutes that have passed since the beginning of the drill. The graph of a is plotted at right. Note that the values are negative since the submarine is below sea level.



(a) What can you say about the tangent line to the graph at the lowest point on the dive?

The tangent line to the graph is horizontal (i.e. it has slope=0) at the lowest point on the dive.

(b) After how many minutes will the submarine reach the lowest point? Round your answer to two decimal places.

Since the tangent line is horizontal at the lowest point, we find a formula for the slope of the tangent line as a function of t; then we set that slope equal to 0 and solve for t to find the time at which the submarine is travelling horizontally. The slope is the derivative of a:

$$a'(t) = -250\sin\left(\frac{t}{2}\right) + 125$$

Now set $a'(t) = 0$ and solve for t:

$$\begin{aligned}
& -250\sin\left(\frac{t}{2}\right) + 125 = 0 \\
\Longrightarrow\ & 250\sin\left(\frac{t}{2}\right) = 125 \\
\Longrightarrow\ & \sin\left(\frac{t}{2}\right) = \frac{1}{2} \\
\Longrightarrow\ & \frac{t}{2} = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \frac{t}{2} = \frac{5\pi}{6} + 2k\pi \quad (k \text{ is any integer}) \\
\Longrightarrow\ & t = \frac{\pi}{3} + 4k\pi \quad \text{or} \quad t = \frac{5\pi}{3} + 4k\pi
\end{aligned}$$

Let's list the values of t we get from these formulas for $k = -1, 0, 1$:

$$\begin{aligned}
t = \frac{\pi}{3} + 4k\pi &: \quad \frac{-11\pi}{3} \approx -11.52, \quad \frac{\pi}{3} \approx 1.05, \quad \frac{13\pi}{3} \approx 13.61 \\
t = \frac{5\pi}{3} + 4k\pi &: \quad \frac{-7\pi}{3} \approx -7.33, \quad \frac{5\pi}{3} \approx 5.24, \quad \frac{17\pi}{3} \approx 17.80
\end{aligned}$$

From the graph we see that our solution for t lies between 0 and 8. The only solutions in this list between 0 and 8 are $t = \frac{\pi}{3} \approx 1.05$ and $t = \frac{5\pi}{3} \approx 5.24$. There are two answers instead of one because this technique also finds the point where a is a maximum. We see from the graph that the time $t = 1.07$ must correspond to the maximum altitude, and the time $t = 5.24$ must correspond to the minimum altitude (i.e. the greatest depth).

Therefore, the submarine reaches its lowest point after $\boxed{t = \frac{5\pi}{3} \approx 5.24 \text{ minutes}}$.

(c) What is the greatest depth the submarine will reach on this dive? Round your answer to the nearest foot.

The minimum altitude of the submarine in feet is

$$
\begin{aligned}
a\left(\frac{5\pi}{6}\right) &= 500\cos\left(\frac{5\pi}{6}\right) + 125\left(\frac{5\pi}{3}\right) - 564 \\
&= 500\left(\frac{-\sqrt{3}}{2}\right) + 125\left(\frac{5\pi}{3}\right) - 564 \\
&= -250\sqrt{3} + 125\left(\frac{5\pi}{3}\right) - 564 \\
&\approx -342.5
\end{aligned}
$$

Therefore, the greatest depth the submarines reaches is approximately $\boxed{343 \text{ feet}}$.

(d) What is the greatest rate at which the submarine's depth will increase during the dive? Include units in your answer.

Let $r(t)$ be the rate of change in depth. Then $r(t) = d'(t)$, where $d(t) = -a(t)$ is the depth of the submarine. Hence we have the formula

$$r(t) = \frac{d}{dt}[-a(t)] = \frac{d}{dt}\left[-500\cos\left(\frac{t}{2}\right) - 125t + 564\right] = 250\sin\left(\frac{t}{2}\right) - 125.$$

We have been asked to find the maximum value of $r(t)$. Following the examples above, we expect that the tangent line at the maximum of $r(t)$ will be horizontal, so we look for solutions of $r'(t) = 0$:

$$
\begin{aligned}
& 0 = r'(t) = 125\cos\left(\frac{t}{2}\right) \\
\Longrightarrow\ & \cos\left(\frac{t}{2}\right) = 0 \\
\Longrightarrow\ & \frac{t}{2} = \frac{\pi}{2} + k\pi \qquad (k \text{ is any integer}) \\
\Longrightarrow\ & t = \pi + 2k\pi.
\end{aligned}
$$

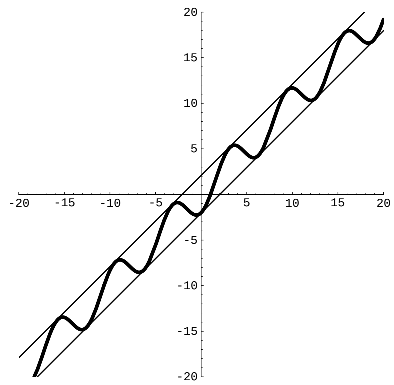
The only solution for t between 0 and 8 is $t = \pi$. Observe that $r(\pi) = 250\sin\left(\frac{\pi}{2}\right) - 125 = 250(1) - 125 = 125$.

We conclude that the maximum valuse of $r(t)$ is 125. Since the units of $d(t) = -a(t)$ are feet and the units of t are minutes, we see that the units of $r(t)$ are $\frac{\text{feet}}{\text{min}}$.

Consequently, the greatest rate of increase of the submarine's depth is $\boxed{125\frac{\text{feet}}{\text{min}}}$.

If your group has finished, start working on this problem from homework #6:

The graphs of $y = x - 2\cos(x)$ and two parallel tangential lines are given below. Find the equations of the two lines.



First, we find the equation of the bottom line because we can immediately identify a point on the curve that it passes through: Observe that the bottom line goes through the point on the curve $y = x - 2\cos(x)$ where $x = 0$. Here, we have $y = 0 - 2\cos(0) = 0 - 2(1) = -2$. So we know that the point $(0, -2)$ is on the line, and it is tangent to the curve there. Thus we can find the slope of the bottom line by calculating the derivative there:

$$\frac{dy}{dx} = 1 + 2\sin(x)$$

$$\Longrightarrow \quad \left.\frac{dy}{dx}\right|_{x=0} = 1 + 2\sin(0) = 1 + 2(0) = 1.$$

That is to say, the lower line has slope 1. We can immediately determine an equation for the lower line using the point-slope formula:

$$y - (-2) = 1(x - 0),$$

that is,

$$y = x - 2.$$

Next, recall that the two lines are parallel, so we know that the upper line also has slope 1. We need to identify a point on this line. Since this line is also tangent wherever it touches the curve, we know that the derivative will equal 1 there, so we solve the equation $\frac{dy}{dx} = 1$:

$$1 + 2\sin(x) = 1$$

$$\Longrightarrow \quad 2\sin(x) = 0$$

$$\Longrightarrow \quad \sin(x) = 0$$

$$\Longrightarrow \quad x = k\pi, \quad \text{where } k \text{ is any integer}$$

From the graph, we see that if k is even, the point on the curve with $x = k\pi$ lies on the bottom line, and if k is odd, the point with $x = k\pi$ lies on the top line. For example, if we take $k = 1$, then we see that the point with $x = \pi$ lies on the top line. Here, $y = \pi - 2\cos(\pi) = \pi - 2(-1) = \pi + 2$. This gives us a point on the top line: $(\pi, \pi + 2)$. Using the fact that the slope of the top line is 1 and the point-slope formula again, we get the equation

$$y - (\pi + 2) = 1(x - \pi),$$

that is

$$y = x + 2.$$

Therefore, the equations of the top and bottom lines are

$$\boxed{y = x + 2} \qquad \text{and} \qquad \boxed{y = x - 2},$$

respectively.