1. (16 total points) Evaluate the following integrals. Simplify your answers where possible.

(a) (8 points) $\int \sqrt{x}e^{\sqrt{x}}\,dx$

Solution: Let $w = \sqrt{x}$, or $w^2 = x$, then since $2w\,dw = dx$ we get

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = \int we^w(2w)\,dw = \int 2w^2e^w\,dw.$$

Integrate by parts with $u = 2w^2$ and $dv = e^w\,dw$:

$$\int 2w^2e^w\,dw = 2w^2e^w - \int 4we^w\,dw.$$

Use parts again with $u = 4w$ and $dv = e^w\,dw$:

$$2w^2e^w - \int 4we^w\,dw = 2w^2e^w - 4we^w + \int 4e^w\,dw = 2w^2e^w - 4we^w + 4e^w + C.$$

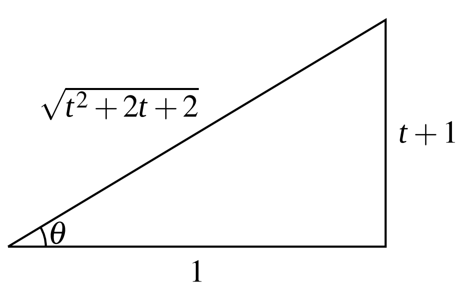
Since $w = \sqrt{x}$,

$$\int \sqrt{x}e^{\sqrt{x}}\,dx = 2w^2e^w - 4we^w + 4e^w + C = 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C.$$

(b) (8 points) $\int \frac{1}{(t^2+2t+2)^{3/2}}\,dt$

Solution:

$$\begin{aligned}\int \frac{1}{(t^2+2t+2)^{3/2}}\,dt &= \int \frac{1}{((t+1)^2+1)^{3/2}}\,dt \\ &= \int \frac{1}{\sec^3\theta}\sec^2\theta\,d\theta \\ &= \int \cos\theta\,d\theta \\ &= \sin\theta + C \\ &= \frac{t+1}{\sqrt{t^2+2t+2}} + C\end{aligned}$$



2. (16 total points) Evaluate the following integrals. Simplify your answers where possible.

(a) (8 points) $\int_0^{\pi} \sin^4\theta \, d\theta$

Solution:

$$\begin{aligned}
\int_0^{\pi} \sin^4\theta \, d\theta &= \int_0^{\pi} \left(\frac{1}{2}(1-\cos 2\theta)\right)^2 d\theta && \text{use } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\
&= \frac{1}{4}\int_0^{\pi} (1-2\cos 2\theta + \cos^2 2\theta) d\theta \\
&= \frac{1}{4}\int_0^{\pi} (1-2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta) d\theta && \text{use } \cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta) \\
&= \frac{1}{4}\int_0^{\pi} (\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta) d\theta \\
&= \frac{1}{4}\left[\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi} \\
&= \frac{1}{4}\left([\frac{3}{2}\pi - 0 + 0] - [0-0+0]\right) \\
&= \frac{3}{8}\pi.
\end{aligned}$$

(b) (8 points) $\int_0^2 x\sqrt{16-x^4}\, dx$

Solution: Solution 1:

$$\begin{aligned}
\int_0^2 x\sqrt{16-x^4}\, dx &= \frac{1}{2}\int_0^4 \sqrt{16-u^2}\, du && \text{sub } u = x^2,\ du = 2x\,dx \\
&= \frac{1}{2}\text{(area of quarter circle of radius 4)} \\
&= \frac{1}{2}\cdot 4\pi = 2\pi.
\end{aligned}$$

Solution 2:

$$\begin{aligned}
\int_0^2 x\sqrt{16-x^4}\, dx &= \frac{1}{2}\int_0^4 \sqrt{16-u^2}\, du && \text{sub } u = x^2,\ du = 2x\,dx \\
&= \frac{1}{2}\int_0^{\pi/2} \sqrt{4^2 - 4^2\sin^2\theta}\, 4\cos\theta \, d\theta && \text{sub } u = 4\sin\theta,\ du = 4\cos\theta\, d\theta \\
&= 8\int_0^{\pi/2} \cos\theta \cdot \cos\theta \, d\theta \\
&= 4\int_0^{\pi/2} (1+\cos 2\theta) d\theta && \text{use } \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \\
&= 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} \\
&= 4\,([\pi/2 + 0] - [0+0]) \\
&= 2\pi
\end{aligned}$$

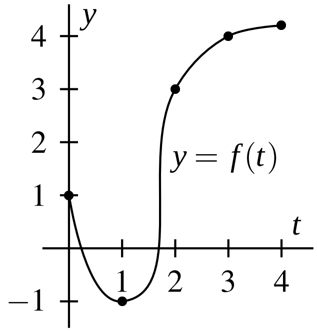
3. (8 points) Evaluate the following integral. Simplify your answer where possible.

$$\int_1^\infty \frac{1+e^x}{e^x(1-e^x)}dx$$

Solution:

$$\begin{aligned}
\int_1^\infty \frac{1+e^x}{e^x(1-e^x)}dx &= \lim_{b\to\infty}\int_1^b \frac{1+e^x}{e^x(1-e^x)}dx \\
&= \lim_{b\to\infty}\int_e^{e^b} \frac{1+u}{u^2(1-u)}\,du \qquad \text{sub } u=e^x,\ du=e^x\,dx \\
&= \lim_{b\to\infty}\int_e^{e^b} \left(\frac{2}{u}+\frac{1}{u^2}+\frac{2}{1-u}\right)du \\
&= \lim_{b\to\infty}\left[2\ln u-\frac{1}{u}-2\ln|1-u|\right]_e^{e^b} \\
&= \lim_{b\to\infty}\left[2\ln\frac{u}{u-1}-\frac{1}{u}\right]_e^{e^b} \\
&= \frac{1}{e}-2\ln\frac{e}{e-1}
\end{aligned}$$

4. (8 points) Suppose that the graph of f is as shown:



Let $G(x) = \int_x^{x^2+x} t\,f(t)\,dt$. Find $G'(1)$.

Solution:

$$G(x) = \int_x^2 t\,f(t)\,dt + \int_2^{x^2+x} t\,f(t)\,dt$$
$$= -\int_2^x t\,f(t)\,dt + \int_2^{x^2+x} t\,f(t)\,dt.$$

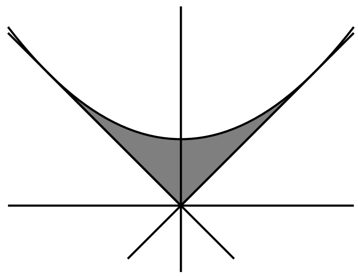
So

$$G'(x) = -x\,f(x) + (2x+1)\cdot(x^2+x)\,f(x^2+x),$$

so

$$G'(1) = -1\cdot f(1) + 3\cdot 2\,f(2)$$
$$= 1 + 3\cdot 2\cdot 3 = 19.$$

5. (12 total points) Let S be the region enclosed by the curves $y = x$, $y = -x$ and $2y = x^2 + 1$.



(a) (6 points) Find the volume of the solid obtained by rotating S about the x-axis.

Solution: Use the washer method (also note that the solid is symmetric with respect the y-axis):

$$\text{Volume} = 2\int_0^1 \left[\pi\left(\frac{x^2+1}{2}\right)^2 - \pi x^2\right] dx = 2\pi \int_0^1 \left[\frac{x^4+2x^2+1}{4} - x^2\right] dx$$
$$= 2\pi \int_0^1 \left[\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{4}\right] dx = 2\pi \left[\frac{x^5}{20} - \frac{x^3}{6} + \frac{x}{4}\right]_{x=0}^1$$
$$= 2\pi\left[\frac{1}{20} - \frac{1}{6} + \frac{1}{4}\right] = 2\pi \frac{3-10+15}{60} = \frac{4\pi}{15}.$$

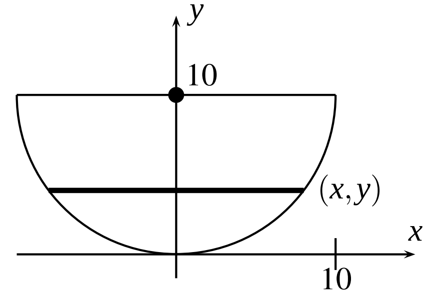
(b) (6 points) Find the volume of the solid obtained by rotating S about the y-axis.

Solution: Use the cylindrical shell method:

$$\text{Volume} = \int_0^1 \left[2\pi x\left(\frac{x^2+1}{2} - x\right)\right] dx = \pi \int_0^1 (x^3 + x - 2x^2) dx$$
$$= \pi\left[\frac{x^4}{4} + \frac{x^2}{2} - \frac{2x^3}{3}\right]_{x=0}^1 = \pi\left[\frac{1}{4} + \frac{1}{2} - \frac{2}{3}\right]$$
$$= \frac{\pi(3+6-8)}{12} = \frac{\pi}{12}.$$

6. (10 points) A tank has the shape of an open-top hemisphere with radius 10 m that is full of water with density 1000 kg/m^3. Set up an integral which computes the work required to empty the tank by pumping all of the water to the top of the tank. DO NOT EVALUATE THIS INTEGRAL.

Solution: Solution 1:



The curve is $x^2 + (y-10)^2 = 10^2$. The incremental work required to remove the slice at height y is

$$\Delta(\text{work}) = \Delta(\text{force}) \cdot \text{distance}.$$

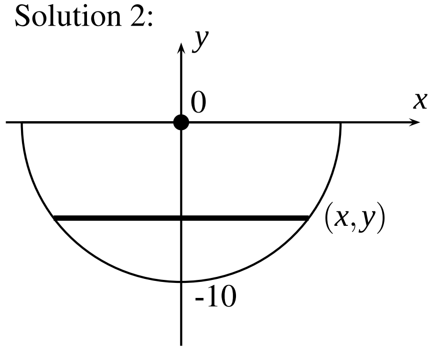
The distance is $10 - y$, and

$$\begin{aligned}\Delta(\text{force}) &= \text{mass} \cdot g = \text{density} \cdot \text{volume} \cdot g \\ &= (1000 \cdot \pi x^2 \cdot \Delta y) \cdot g \\ &= 1000 g\pi[10^2 - (y-10)^2]\Delta y.\end{aligned}$$

So

$$\begin{aligned}\text{Work} = \int_0^{10} \Delta(\text{work}) &= \int_0^{10} 1000 g\pi[100 - (y^2 - 20y + 100)]\, dy \\ &= \int_0^{10} 1000 g\pi(20y - y^2)(10-y)\, dy.\end{aligned}$$

Solution 2:



The curve is $x^2 + y^2 = 10^2$, with $y < 0$. The incremental work required to remove the slice at height y is

$$\Delta(\text{work}) = \Delta(\text{force}) \cdot \text{distance}.$$

The distance is $-y$, and

$$\begin{aligned}\Delta(\text{force}) &= \text{mass} \cdot g = (1000 \cdot \pi x^2 \cdot \Delta y) \cdot g \\ &= 1000 g\pi[100 - y^2]\Delta y.\end{aligned}$$

So

$$\begin{aligned}\text{Work} &= \int_{-10}^{0} \Delta(\text{work}) \\ &= \int_{-10}^{0} 1000g\pi(100-y^2)(-y)\,dy.\end{aligned}$$

7. (8 total points)

(a) (4 points) Set up but DO NOT EVALUATE an integral to compute the arc length of the curve $y = \sin^2(\pi x)$, for $0 \le x \le 1$.

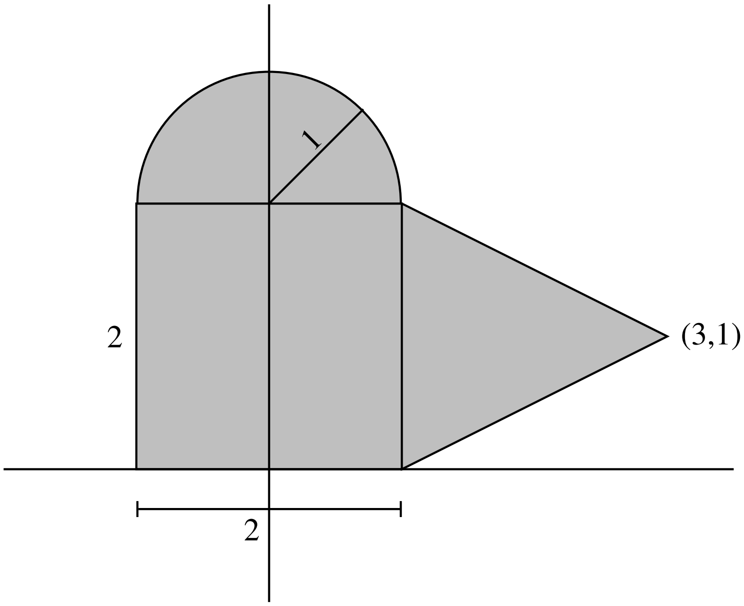
Solution: $y = \sin^2(\pi x)$, so $y' = 2\pi \sin(\pi x) \cdot \cos(\pi x) = \pi \cdot \sin(2\pi x)$.
So the arc length is $L = \int_0^1 \sqrt{1+\pi^2 \sin^2(2\pi x)}\,dx$.

(b) (4 points) Approximate the length of the above curve via Simpson's rule with $n = 4$. SIMPLIFY THE SUM, but LEAVE YOUR ANSWER IN EXACT FORM.

Solution:

$$\begin{aligned}L \cong S_4 &= \frac{1}{12}\left[f(0)+4f(\tfrac{1}{4})+2f(\tfrac{1}{2})+4f(\tfrac{3}{4})+f(1)\right] \\ &= \frac{1}{12}\left[1+4\sqrt{1+\pi^2}+2+4\sqrt{1+\pi^2}+1\right] \\ &= \frac{1+2\sqrt{1+\pi^2}}{3}.\end{aligned}$$

8. (10 points) Find the x-coordinate of the centroid of the shaded region below.



Solution: Let region I be the semicircle, region II be the square, region III be the triangle. The top line defining the triangle is given by the equation $y = -\frac{x}{2} + \frac{5}{2}$; the bottom line is $y = \frac{x}{2} - \frac{1}{2}$.

Let M_y denote the moment about the y-axis. Then

$$\begin{aligned} M_y(I) &= 0 \\ M_y(II) &= 0 \\ M_y(III) &= \int_1^3 x\left[(-\frac{x}{2}+\frac{5}{2}) - (\frac{x}{2}-\frac{1}{2})\right] dx \\ &= \int_1^3 x(3-x)\,dx = \int_1^3 (3x - x^2)\,dx \\ &= \frac{3x^2}{2} - \frac{x^3}{3}\bigg|_1^3 = \frac{27}{2} - 9 - \frac{3}{2} + \frac{1}{3} = \frac{10}{3}. \end{aligned}$$

The x-coordinate of the center of mass is given by

$$\frac{M_y(I)+M_y(II)+M_y(III)}{\text{area } I + \text{area } II + \text{area } III} = \frac{10/3}{\pi/2+4+2} = \frac{20}{3(\pi+12)}.$$

9. (12 total points) A tank contains 100 liters of fresh water. Water containing s grams of salt per liter enters the tank at the rate of 5 liters/minute, and the well-mixed solution leaves at the same rate.

   (a) (6 points) Write down a differential equation for the amount of salt in the tank at time t. (This equation will contain s.)

   Solution: Let $m(t)$ be the amount of salt (in grams) in the tank at time t. Then

   $$\frac{dm}{dt} = 5s - \frac{m(t)}{100} \cdot 5 = 5s - \frac{m(t)}{20}.$$

   (b) (6 points) Suppose that after 10 minutes, the concentration of salt in the tank is 3 grams/liter. Find s.

   Solution:

   $$\frac{dm}{dt} = \frac{1}{20}(100s - m)$$
   $$\int \frac{dm}{100s - m} = \int \frac{1}{20} dt$$
   $$-\ln|100s - m| = \frac{t}{20} + C$$
   $$|100s - m| = Ae^{-t/20}, A \text{ positive constant}$$

   At $t = 0$:

   $$100s = Ae^0 = A.$$

   Therefore,

   $$100s - m(t) = 100s \cdot e^{-t/20},$$
   $$m(t) = 100s(1 - e^{-t/20}).$$

   $m(10) = 3 \cdot 100 = 300$, so

   $$300 = 100s(1 - e^{-1/2})$$
   $$s = \frac{3}{(1 - e^{-1/2})} = 7.62.$$