Math 125 **Final Examination** **Spring 2006**

Your Name

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (both sides). Do not share notes.
- Give your answers in exact form, except as noted in particular problems.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. You may use any of the 20 integrals from the table on p. 506 of the text without deriving them. Show your work in evaluating any other integrals, even if they are on your note sheet.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	14	
2	14	
3	8	
4	8	
5	8	
6	8	

Question	Points	Score
7	8	
8	8	
9	8	
10	8	
11	8	
Total	100	

1. (14 total points) Evaluate the following integrals.

   (a) (7 points) $\displaystyle\int \frac{x}{\sqrt{8-2x-x^2}}\,dx$

Solution: First, observe that $8-2x-x^2 = 9-1-2x-x^2 = 3^2-(x+1)^2$. Therefore, we can rewrite the above integral as

$$\int \frac{x}{\sqrt{3^2-(x+1)^2}}dx.$$

Making the substitution $u = x+1$, this integral becomes

$$\int \frac{u-1}{\sqrt{3^2-u^2}}du.$$

This integral can be split into two parts:

$$= \int \frac{u}{\sqrt{3^2-u^2}}du - \int \frac{du}{3^2-u^2}.$$

We evaluate the first term by making the substitution $v = 9-u^2$, and the second integral is one of the integrals appearing on p. 506 of the book. In total, we get

$$= -\sqrt{9-(x+1)^2} - \sin^{-1}(\frac{x+1}{a}) + C$$

   (b) (7 points) $\displaystyle\int \ln(\sec(x))\sec(x)\tan(x)\,dx$

Solution: *substitution*: Let $t = \sec(x)$, so $dt = \sec(x)\tan(x)dx$. We obtain

$$\int \ln(\sec(x))\sec(x)\tan(x)\ dx = \int \ln(t)\ dt.$$

integration by parts set up:
Let $u = \ln(t)$ and $dv = dt$, so $du = \frac{1}{t}dt$ and $v = t$.
using integration by parts formula:

$$\begin{aligned}\int \ln(t)\ dt &= t\ln(t) - \textstyle\int t\frac{1}{t}\ dt \\ &= t\ln(t) - t + C \\ &= \sec(x)\ln(\sec(x)) - \sec(x) + C\end{aligned}$$

2. (14 total points) Evaluate the following integrals. Leave your answers in exact form: do not use decimal expansions.

   (a) (7 points) $\int_0^{\pi/4} \sin^3(2x)\cos^2(2x)\,dx$

Solution:

$$\begin{aligned}
\int_0^{\pi/4} \sin^3(2x)\cos^2(2x)\;dx &= \int_0^{\pi/4} \sin^2(2x)\cos^2(2x)\sin(2x)\;dx \\
&= \int_0^{\pi/4} (1-\cos^2(2x))\cos^2(2x)\sin(2x)\;dx \\
&= -\frac{1}{2}\int_1^0 (1-u^2)u^2\;du, \text{ where } u=\cos(2x) \\
&= \frac{1}{2}\int_0^1 (u^2-u^4)\;du \\
&= \frac{1}{2}\left[\frac{1}{3}u^3-\frac{1}{5}u^5|_0^1\right] \\
&= \frac{1}{2}\left[(\frac{1}{3}(1)^3-\frac{1}{5}(1)^5)-(\frac{1}{3}(0)^3-\frac{1}{5}(0)^5)\right] \\
&= \frac{1}{15} = 0.0\overline{6}
\end{aligned}$$

   (b) (7 points) $\int \frac{\cos x}{4-\sin^2 x}\,dx$

Solution: Use the substitution $u=\sin x$, so that $du=\cos x\,dx$. This gives

$$\int \frac{\cos x}{4-\sin^2 x}\,dx = \int \frac{1}{4-u^2}\,du = -\int \frac{1}{(u-2)(u+2)}\,du.$$

Using partial fractions this becomes

$$\begin{aligned}
\int \frac{1}{4-u^2}\,du &= -\int\left(\frac{1}{4}\frac{1}{u-2}-\frac{1}{4}\frac{1}{u+2}\right)du \\
&= -\left(\frac{1}{4}\ln|u-2|-\frac{1}{4}\ln|u+2|\right)+C \\
&= \boxed{\frac{1}{4}\ln|\sin x+2|-\frac{1}{4}\ln|\sin x-2|+C.}
\end{aligned}$$

3. (8 points) Evaluate the integral

$$\int_2^\infty x^5 e^{-x^3}\,dx.$$

Solution: **Solution 1** (substitution, then integration by parts):

$$\int_2^\infty x^5 e^{-x^3}\,dx = \lim_{t\to\infty}\int_2^t x^5 e^{-x^3}\,dx$$

$$\int_2^t x^3 \cdot x^2 e^{-x^3}\,dx \qquad \text{(substitution: } w = x^3\text{, so } dw = 3x^2 dx\text{)}$$

$$= \frac{1}{3}\int_8^{t^3} we^{-w}\,dw$$

(by parts: $u = w$, $dv = e^{-w}dw$, so $du = dw$, $v = -e^{-w}$)

$$= \frac{1}{3}\left[-we^{-w}\Big|_8^{t^3} + \int_8^{t^3} e^{-w}\,dw\right]$$

$$= \frac{1}{3}\left[-t^3e^{-t^3} + 8e^{-8} - e^{-w}\Big|_8^{t^3}\right]$$

$$= \frac{1}{3}\left[-t^3e^{-t^3} + 8e^{-8} - e^{-t^3} + e^{-8}\right]$$

$$= \frac{1}{3}\left[9e^{-8} - (t^3+1)e^{-t^3}\right]$$

So

$$\int_2^\infty x^5 e^{-x^3}\,dx = \lim_{t\to\infty}\frac{1}{3}\left[9e^{-8} - (t^3+1)e^{-t^3}\right] = \frac{1}{3}\left[9e^{-8}+0\right] = \boxed{3e^{-8}}$$

Solution 2 (integration by parts first):

$$\int_2^\infty x^5 e^{-x^3}\,dx = \lim_{t\to\infty}\int_2^t x^5 e^{-x^3}\,dx$$

$$\int_2^t x^3 \cdot x^2 e^{-x^3}\,dx \qquad \text{(by parts: } u = x^3,\ dv = x^2e^{-x^3}dx\text{, so } du = 3x^2dx,\ v = -\frac{1}{3}e^{-x^3}\text{)}$$

$$= -\frac{1}{3}x^3e^{-x^2}\Big|_2^t + \int_2^t x^2e^{-x^3}dx$$

(for integral, substitute or guess and check)

$$= -\frac{t^3}{3}e^{-t^3} + \frac{8}{3}e^{-8} - \frac{1}{3}e^{-x^3}\Big|_2^t$$

$$= -\frac{t^3}{3}e^{-t^3} + \frac{8}{3}e^{-8} - \frac{1}{3}e^{-t^3} + \frac{1}{3}e^{-8}$$

So

$$\int_2^\infty x^5 e^{-x^3}\,dx = \lim_{t\to\infty}\left(3e^{-8} - \frac{t^3+1}{3}e^{-t^3}\right) = \boxed{3e^{-8}}.$$

4. (8 points) A particle is moving along a straight line with acceleration $a(t) = 2t$. At time $t = 0$, its velocity is $v_0 = -4$. What is the *total distance* traveled by the particle from time $t = 0$ to time $t = 3$?

Solution:

$$\begin{aligned}
v(t) &= \int a(t)\,dt = \int 2t\,dt = t^2 + C \\
-4 &= v(0) = 0^2 + C \;\Rightarrow\; C = -4 \\
v(t) &= t^2 - 4 \\
\text{Total distance} &= \int_0^3 |v(t)|\,dt \\
v(t) &= (t-2)(t+2), \quad \text{positive for } 2 < t \le 3, \text{ negative for } 0 \le t < 2 \\
\text{Tot dist} &= \int_0^2 -v(t)\,dt + \int_2^3 v(t)\,dt \\
&= \int_0^2 (4 - t^2)dt + \int_2^3 (t^2 - 4)dt \\
&= \left(4t - \frac{1}{3}t^3\right)\Big|_0^2 + \left(\frac{1}{3}t^3 - 4t\right)\Big|_2^3 \\
&= \left[\left(8 - \frac{8}{3}\right) - 0\right] + \left[(-3) - \left(\frac{8}{3} - 8\right)\right] = \boxed{\frac{23}{3}}
\end{aligned}$$

5. (8 points) Approximate the integral $\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx$ by using the trapezoidal rule with $n = 4$. Express your answer as a decimal.

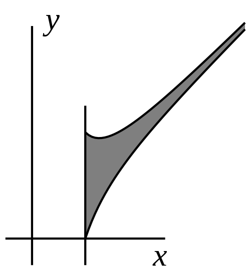
Solution: Breaking the interval $[0,1]$ into 4 equally spaced sub-intervals, the width Δx of each sub-interval is $\frac{1}{4}$. The trapezoidal rule then tells us:

$$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}dx \sim \frac{1}{\sqrt{2\pi}}\frac{\frac{1}{4}}{2}\left(e^0 + 2e^{-\frac{1}{32}} + 2e^{-\frac{1}{8}} + 2e^{-\frac{9}{32}} + e^{-\frac{1}{2}}\right) \sim .34008184$$

6. (8 points) Consider the *unbounded* region S contained within the curves

$$y = x + \frac{1}{x^2}, \qquad y = x - \frac{1}{x^2} \qquad \text{and} \qquad x = 1$$

as shown in the picture below.



Is the area of S finite or infinite? If it is finite, justify your conclusion and find this area. If it is infinite, carefully explain why.

Solution: The Area is given by the improper integral

$$A = \int_1^\infty \left[\left(x + \frac{1}{x^2}\right) - \left(x - \frac{1}{x^2}\right)\right] dx = \int_1^\infty \frac{2}{x^2}\, dx.$$

This last integral is convergent and its value is given by

$$\int_1^\infty \frac{2}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{2}{x^2}\, dx = \lim_{t\to\infty} \left[-\frac{2}{x}\right]\Big|_1^t = 2.$$

So the Area is finite and equal to 2.

7. (8 points) Let R be the region below the curve $y = \frac{1}{x}$, above the x-axis, and between the vertical lines $x = 1$ and $x = 3$. Set up and evaluate a definite integral for the volume of the solid obtained by rotating R about the vertical line $x = -2$.

Solution: A typical cylindrical shell has 'radius' = $x + 2$ and 'height' = $1/x$. Thus,

$$V = \int_1^3 2\pi(x+2)\frac{1}{x}\,dx = 2\pi\int_1^3 1 + \frac{2}{x}\,dx$$

$$= 2\pi\left[x + 2\ln(x)|_1^3\right] = 2\pi[(3 + 2\ln(3)) - (1 + 2\ln(1))] = 4\pi[1 + \ln(3)]$$

8. (8 points) A small circular pool has a radius of 10 ft, the sides are 3 ft high, and the depth of the water is 2 ft. How much work (in ft-lb) is required to pump all of the water out over the side of the pool? (Water weighs 62.5 lb/ft^3.)

Solution: Let y be the distance (in feet) from the bottom of the pool, so the water starts in the interval $0 \le y \le 2$. Divide this interval $[0,2]$ into n equal subintervals:

$$\Delta y = \frac{2-0}{n} = \frac{2}{n},$$
$$y_i = 0 + i\Delta y \quad \text{for } 0 \le i \le n$$
$$y_0 = 0,\ y_1 = \Delta y, \ldots,\ y_i = i\Delta y, \ldots,\ y_n = 2$$

For $1 \le i \le n$, let

W_i = work required to move layer of water between y_{i-1} and y_i to $y = 3$ (in ft-lb)

V_i = volume of this layer (in ft^3) $= \pi 10^2 \Delta y$

F_i = weight of this layer (in lb) $= (62.5\frac{\text{lb}}{\text{ft}^3})(V_i \text{ft}^3)$

d_i = distance this layer must move (in ft) against the force of gravity $\approx 3 - y_i$

Then

$$W_i = F_i d_i \approx (62.5 V_i \text{lb})(3 - y_i \text{ft})$$
$$= (62.5)(\pi 10^2 \Delta y)(3 - y_i)\text{ft-lb}.$$

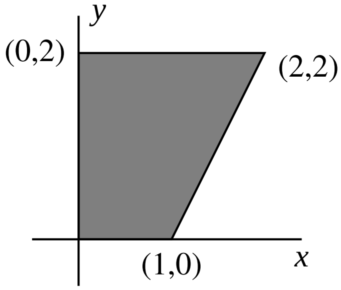
So

$$\text{Total work} = W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 6250\pi(3 - y_i)\Delta y.$$

By Riemann sum argument,

$$W = \int_0^2 6250\pi(3-y)\,dy$$
$$= 6250\pi(3y - \frac{1}{2}y^2)\Big|_0^2$$
$$= 6250\pi(4-0) = \boxed{25{,}000\pi \text{ ft-lb}} \approx 78{,}540 \text{ ft-lb}.$$

9. (8 points) Find the x-coordinate $\bar{x}$ of the center of mass of the region below.



Solution: Note: there are many correct ways to do this problem

First we compute the area A of the region. Since it is composed of a rectangle with side lengths 1 and 2 and a right triangle of height 2 and base 1, we see that

$$A = (2)(1) + \frac{1}{2}(2)(1) = 3.$$

We then may use the formula

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\,dx.$$

Here since $f(x) = 2$ for $0 \le x \le 2$ but $g(x) = 0$ for $0 \le x \le 1$ and $g(x) = 2x - 2$ for $1 \le x \le 2$, we have

$$\begin{aligned}
\bar{x} &= \frac{1}{3}\left[\int_0^1 2x\,dx + \int_1^2 x(2-(2x-2))\,dx\right] \\
&= \frac{1}{3}\left[\int_0^1 2x\,dx + \int_1^2 (4x - 2x^2)\,dx\right] \\
&= \frac{1}{3}\left[x^2|_0^1 + (2x^2 - \frac{2}{3}x^3)|_1^2\right] \\
&= \frac{1}{3}\left[1 + (8 - \frac{2}{3}8) - (2 - \frac{2}{3})\right] \\
&= \frac{7}{9}.
\end{aligned}$$

So $\bar{x} = \frac{7}{9}$.

10. (8 points) Find the solution $y(x)$ for $x \geq 1$ of the initial value problem

$$\frac{y}{x^3}\frac{dy}{dx} = 4\ln(x) \ , \quad y(1) = 2.$$

Solution: *separating variables*: $ydy = 4x^3\ln(x)dx$

integration: $\frac{1}{2}y^2 = 4\int x^3\ln(x)dx$

integration by parts: $u = \ln(x)$ and $dv = x^3dx$ give

$$\frac{1}{2}y^2 = 4\frac{1}{4}x^4\ln(x) - 4\frac{1}{4}\int x^4\frac{1}{x}\ dx$$

$$\frac{1}{2}y^2 = x^4\ln(x) - \frac{1}{4}x^4 + C$$

$$y = \pm\sqrt{2x^4\ln(x) - \frac{1}{2}x^4 + 2C}$$

initial condition:

$$2 = +\sqrt{2(1)^4\ln(1) - \frac{1}{2}(1)^4 + 2C}$$

$$4 = 2C - \frac{1}{2} \Rightarrow 2C = 4.5$$

final solution:

$$y = \sqrt{2x^4\ln(x) - \frac{1}{2}x^4 + 4.5}$$

11. (8 total points) Suppose we have a colony of bacteria living in a Petri dish. Due to space limitations, there is a maximum number, k, of bacteria that can live in the dish. Let $P(t)$ be the population of the bacterial colony at time t. According to one model for population growth, the rate of growth of the population $\frac{dP}{dt}$ is proportional to the difference of the threshold population k and the present population; in other words

$$\frac{dP}{dt} = c(k-P).$$

The constant of proportionality c measures how quickly the bacteria multiply. For simplicity, we take $c = 1$.

(a) (6 points) Solve this differential equation for the unknown function $P(t)$.

Solution: We use the technique of separation of variables. This gives

$$\int \frac{dP}{k-P} = \int dt.$$

Evaluating both sides, we get $-\ln(k-P) = t + c'$, or $k - P = ce^{-t}$ and hence

$$P = k - ce^{-t}.$$

(b) (2 points) If $k = 5,000,000$ and the initial population size is $P(0) = 1,000,000$, compute $\lim_{t\to\infty} P(t)$.

Solution: From the form of the solution in the previous problem, we see that $\lim_{t\to\infty} P(t) = k$, independent of c. Anyway, we plug $P(0) = 1,000,000$ and $k = 5,000,000$ into the solution we obtained in part (a). This gives us

$$1,000,000 = 5,000,000 - ce^0.$$

In other words, $c = 4,000,000$. Therefore, the solution satisfying these initial conditions is

$$P(t) = 5,000,000 - (4,000,000)e^{-t}.$$

Therefore, the limit as $t \to \infty$ is $5,000,000$.