Math 125 **Final Examination** **Winter 2006**

Your Name

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (both sides). Do not share notes.
- Give your answers in exact form, except as noted in particular problems.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. You may use any of the 20 integrals from the table on p. 506 of the text without deriving them. Show your work in evaluating any other integrals, even if they are on your note sheet.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 12 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	14	
2	14	
3	7	
4	8	
5	8	
6	7	

Question	Points	Score
7	8	
8	8	
9	8	
10	10	
11	8	
Total	100	

1. (14 total points) Evaluate the following integrals.

(a) (7 points) $\displaystyle\int \frac{dx}{x^3+x^2}$

Solution: Partial Fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

So

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

Set $x=0$ to get $B=1$; set $x=-1$ to get $C=1$; set say $x=1$ to get $A=-1$. Then

$$\int \frac{dx}{x^3+x^2} = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx$$
$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

(b) (7 points) $\displaystyle\int \frac{(x+7)\,dx}{x^2+6x+13}$

Solution: Complete the square:

$$\int \frac{x+7}{x^2+6x+13}\,dx = \int \frac{x+7}{(x+3)^2+4}\,dx$$

Now let $u = x+3$ to get

$$\int \frac{u+4}{u^2+4}\,du = \frac{1}{2}\ln(u^2+4) + \frac{4}{2}\tan^{-1}\frac{u}{2} + C$$
$$= \frac{1}{2}\ln(x^2+6x+13) + 2\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

2. (14 total points) Evaluate the following integrals. Leave your answers in exact form: do not use decimal expansions.

   (a) (7 points) $\displaystyle\int_0^{\sqrt{5}} \frac{x^3\,dx}{\sqrt{9-x^2}}$

Solution: There are several ways to do this integral, I'll list the two likely methods.

<u>u-sub: $u = 9 - x^2$</u>

$$\int_0^{\sqrt{5}} \frac{x^3\,dx}{\sqrt{9-x^2}} = -\frac{1}{2}\int_9^4 \frac{9-u}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int_4^9 \frac{9}{\sqrt{u}} - \sqrt{u}\,du$$

$$= \left[9\sqrt{u} - \frac{1}{3}u^{3/2}\right]_4^9$$

$$= (27-9) - (18 - \tfrac{8}{3}) = \tfrac{8}{3}$$

<u>trig sub: $x = 3\sin\theta$</u>

$$\int \frac{x^3\,dx}{\sqrt{9-x^2}} = 27\int \sin^3\theta\,d\theta$$

$$= 27\int (1-\cos^2\theta)\sin\theta\,d\theta$$

$$= -27\int (1-u^2)\,du \qquad (u = \cos\theta)$$

$$= -27\left(u - \tfrac{1}{3}u^3\right) = -27(\cos\theta - \tfrac{1}{3}\cos^3\theta)$$

$$= -9\sqrt{9-x^2} + \tfrac{1}{3}(9-x^2)^{3/2}$$

Plug in limits to get $(-18+\frac{8}{3}) - (-27+9) = \frac{8}{3}$.

   (b) (7 points) $\displaystyle\int_1^4 \frac{\tan^{-1}(\sqrt{t})}{\sqrt{t}}\,dt$ (Recall that $\tan^{-1} = \arctan$.)

Solution:
Make the substitution

$$u = \sqrt{t}\ ,$$

so that

$$du = \frac{dt}{2\sqrt{t}}\ .$$

$$
\begin{aligned}
\int_1^4 \frac{\tan^{-1}(\sqrt{t})}{\sqrt{t}}\,dt &= \int_1^2 \tan^{-1}(u)(2\,du) \\
&= 2\left(u\tan^{-1}(u) - \frac{1}{2}\ln(1+u^2)\right|_1^2 \\
&= 2\left[(2\tan^{-1}(2) - \frac{1}{2}\ln(5)) - (\tan^{-1}(1) - \frac{1}{2}\ln(2))\right] \\
&= 4\tan^{-1}(2) - (\ln(5) - \ln(2)) - 2\tan^{-1}(1) \\
&= 4\tan^{-1}(2) - \ln\frac{5}{2} - \frac{\pi}{2}
\end{aligned}
$$

3. (7 points) Evaluate the following integral.

$$\int \frac{x}{(x^2+2x-3)^{3/2}}\,dx$$

Solution: First, complete the square to get $\displaystyle\int \frac{x}{[(x+1)^2-4]^{3/2}}\,dx$

I'd next do the substitution $t = x+1$, $dt = dx$: $\displaystyle\int \frac{t-1}{(t^2-4)^{3/2}}\,dt$

Then separate into $\displaystyle\int \frac{t}{(t^2-4)^{3/2}}\,dt - \int \frac{1}{(t^2-4)^{3/2}}\,dt$

The first integrates easily to $\displaystyle\frac{-1}{\sqrt{x^2+2x-3}}+C_1$

To solve the second, make the substitution $t = 2\sec\theta$, $dt = 2\sec\theta\tan\theta\,d\theta$

This gives $\displaystyle\int \frac{\cos\theta}{4\sin^2\theta}\,d\theta$

This integrates to $\displaystyle\frac{-1}{4\sin\theta}+C_2 = -\frac{x+1}{4\sqrt{x^2+2x-3}}+C_2$

The complete answer is $\displaystyle\frac{x-3}{4\sqrt{x^2+2x-3}}+C$

4. (8 points) Determine whether the following improper integral is convergent or divergent. Evaluate it if it is convergent.

$$\int_0^\infty x^3 e^{-x^2}\,dx$$

Solution: Use l'Hopital after computing

$$F(b) = \int_0^b x^3 e^{-x^2}\,dx = \frac{1}{2}\int_0^{b^2} ue^{-u}\,du$$
$$= -\frac{1}{2}ue^{-u}|_0^{b^2} + \frac{1}{2}\int_0^{b^2} e^{-u}\,du$$
$$= \frac{1}{2}\left(1 - e^{-b^2} - b^2e^{-b^2}\right)$$

Since $\lim_{b\to\infty} e^{-b^2} = 0$ and by l'Hopital,

$$\lim_{b\to\infty} b^2e^{-b^2} = \lim_{b\to\infty}\frac{b^2}{e^{b^2}} = \lim_{b\to\infty}\frac{2b}{2be^{b^2}} = \lim_{b\to\infty}\frac{1}{e^{b^2}} = 0,$$

we have that

$$\int_0^\infty x^3 e^{-x^2}\,dx = \lim_{b\to\infty} F(b) = \frac{1}{2}.$$

5. (8 total points) An object is moving along the x-axis, with acceleration at time $t \geq 0$ given by $a(t) = \dfrac{-2000}{(t+2)^3}$ ft/sec^2. At time $t = 0$, its velocity is 240 ft/sec.

(a) (4 points) At what time $t \geq 0$ does the object reverse direction?

Solution: Integrate to find $v(t)$:

$$v(t) = \int \frac{-2000}{(t+2)^3}\,dt + C = \frac{1000}{(t+2)^2} + C$$

Use $v(0) = 240$ to get $C = -10$, so $v(t) = \frac{1000}{(t+2)^2} - 10$. Now set $v(t) = 0$ and solve for t: $(t+2)^2 = 100$, so $t+2 = 10$, so

$$t = 8 \text{ sec.}$$

(b) (4 points) Find the total distance traveled by the object from $t = 0$ to $t = 18$ sec.

Solution: Total distance is

$$\begin{aligned}
\int_0^{18} |v(t)|\,dt &= \int_0^8 \left(\frac{1000}{(t+2)^2} - 10\right) dt - \int_8^{18} \left(\frac{1000}{(t+2)^2} - 10\right) dt \\
&= \left(-\frac{1000}{t+2} - 10t\right)\Bigg|_0^8 + \left(\frac{1000}{t+2} + 10t\right)\Bigg|_8^{18} \\
&= (-100 - 80 + 500) + (50 + 180 - 100 - 80) \\
&= 370 \text{ feet}
\end{aligned}$$

6. (7 points) Let R be the region enclosed by the curves

$$y = |x|, \quad y = x^2 - 2.$$

Sketch R and find its area.

Solution: For $x > 0$, $x = x^2 - 2$, so $0 = x^2 - x - 2 = (x-2)(x+1)$, which imples $x = 2$.

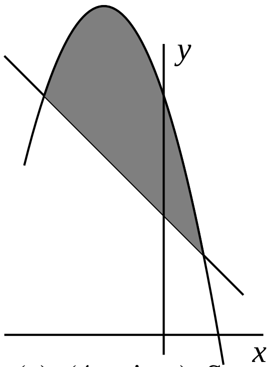
set up the integral:

$$\int_{-2}^{2} (|x| - (x^2 - 2))\, dx\ .$$

By symmetry,

$$\begin{aligned}
\int_{-2}^{2} (|x| - (x^2 - 2))\, dx &= 2\int_{0}^{2} (x - (x^2 - 2))\, dx \\
&= 2\int_{0}^{2} (x - x^2 + 2)\, dx \\
&= 2\left(\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right|_0^2 \\
&= 2\left(2 - \frac{8}{3} + 4\right) \\
&= \frac{20}{3}\ .
\end{aligned}$$

7. (8 total points) Let R be the region bounded by $y = -x^2 - 3x + 6$ and $x + y - 3 = 0$; see the picture.



(a) (4 points) Set up an integral for the volume obtained by rotating R about the vertical line $x = 3$. DO NOT EVALUATE THE INTEGRAL.

Solution: Find intersection points: $-x^2 - 3x + 6 = 3 - x$
$0 = x^2 + 2x - 3$ so $x = 1, -3$.

$$\int_{-3}^{1} \Big((-x^2 - 3x + 6) - (3 - x) \Big) 2\pi(3 - x)dx$$

(b) (4 points) Set up an integral for the volume obtained by rotating R about the horizontal line $y = 0$. DO NOT EVALUATE THE INTEGRAL.

Solution:

$$\int_{-3}^{1} \pi \Big((-x^2 - 3x + 6)^2 - (3 - x)^2 \Big) dx$$

8. (8 total points)

   (a) (4 points) Set up but DO NOT EVALUATE an integral to compute the arc length of the curve $y = x^3$, for $0 \leq x \leq 1$.

   Solution: $y' = 3x^2$

   $$\int_0^1 \sqrt{1+9x^4}\,dx.$$

   (b) (4 points) Approximate the length of the above curve using the trapezoidal rule with $n = 5$. Do not simplify the sum: leave your answer in exact form.

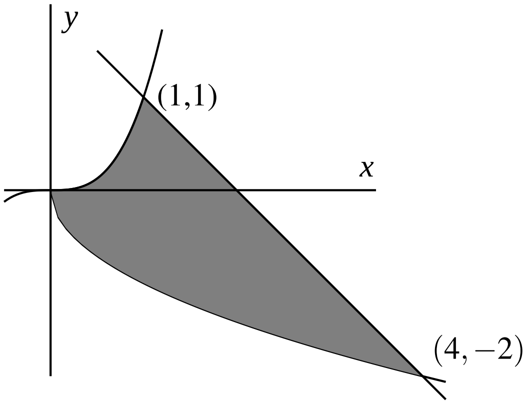
   Solution: $\Delta x = \frac{1-0}{5} = .2$

   $$T_5 = (.2)\left(\frac{1}{2}(1+\sqrt{10}) + \sqrt{1+9(.2)^4} + \sqrt{1+9(.4)^4} + \sqrt{1+9(.6)^4} + \sqrt{1+9(.8)^4}\right)$$

9. (8 points) Consider the region bounded by the curves

$$y = x^3, \quad x + y = 2, \quad y = -\sqrt{x}.$$

The area of this region is $49/12$. Find the x-coordinate of its center of mass. Leave your answer in exact form: do not use decimal expansions.



Solution:

$$\overline{x} = \frac{12}{49}\int_0^4 x\left[f(x) - g(x)\right] dx$$

$$= \frac{12}{49}\left[\int_0^1 x(x^3 + \sqrt{x})\,dx + \int_1^4 x(2 - x + \sqrt{x})\,dx\right]$$

$$= \frac{12}{49}\left[\int_0^1 x^4 + x^{3/2}\,dx + \int_1^4 2x - x^2 + x^{3/2}\,dx\right]$$

$$= \frac{12}{49}\left[\left(\tfrac{1}{5}x^5 + \tfrac{2}{5}x^{5/2}\right)\Big|_0^1 + \left(x^2 - \tfrac{1}{3}x^3 + \tfrac{2}{5}x^{5/2}\right)\Big|_1^4\right]$$

$$= \frac{12}{49}\left[\left(\frac{1}{5} + \frac{2}{5}\right) + \left(4^2 - \frac{1}{3}4^3 + \frac{2}{5}4^{5/2}\right) - \left(1 - \frac{1}{3} + \frac{2}{5}\right)\right]$$

$$= \frac{12}{49}\left[\left(\frac{1}{5} + \frac{2}{5}\right) + \left(16 - \frac{64}{3} + \frac{64}{5}\right) - \left(1 - \frac{1}{3} + \frac{2}{5}\right)\right]$$

$$= \frac{12}{7}$$

10. (10 points) At time $t = 0$, a tank contains 100 gallons of pure gasoline. A mixture whose volume is 30% ethanol and 70% gasoline is pumped into the tank at a rate of 2 gallons per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. Find a formula for the number of gallons of ethanol in the tank after t minutes.

Solution: Let y be the nunber of gallons of tasty ethanol in the tank after t minutes.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$(\text{rate in}) = \left(2\,\frac{\text{Gal}}{\text{min}}\right) \cdot \left(0.3\,\frac{\text{Gal Ethanol}}{\text{Gal}}\right) = 0.6\,\frac{\text{Gal Ethanol}}{\text{min}}$$

$$(\text{rate out}) = \left(\frac{y(t)}{100}\,\frac{\text{Gal Ethanol}}{\text{Gal}}\right) \cdot \left(2\,\frac{\text{Gal}}{\text{min}}\right) = \frac{y(t)}{50}\,\frac{\text{Gal Ethanol}}{\text{min}}$$

$$\frac{dy}{dt} = 0.6 - \frac{y(t)}{50} = \frac{30-y}{50}$$

$$\int \frac{dy}{30-y} = \int \frac{dt}{50}$$

$$-\ln|30-y| = \frac{t}{50} + C$$

Since $y = 0$ when $t = 0$, we get $C = -\ln 30$ and $|30-y| = 30e^{-t/50}$

Since $30 - y \geq 0$, the answer is $y = 30 - 30e^{-t/50}$

11. (8 points) Find the function $y(x)$ which satisfies $\dfrac{dy}{dx} = \dfrac{x(y^2+1)}{\sqrt{x^2-1}}$ such that $y = 1$ when $x = \sqrt{2}$.

Solution:

$$\frac{dy}{y^2+1} = \frac{x}{\sqrt{x^2-1}}\,dx,$$

thus

$$\arctan y + C = \int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}$$

and

$$\arctan 1 + C = \frac{\pi}{4} + C = 1 \text{ and } C = 1 - \frac{\pi}{4}.$$

Hence

$$y = \tan\left(\sqrt{x^2-1} + \frac{\pi}{4} - 1\right).$$