Math 125 **Midterm 1 Solutions** **Burdzy, Autumn 2015**

Your Name

Your Signature

Student ID #

Circle quiz section and print TA's name:
EA EB EC ED FA FB FC FD

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (both sides).
- You can use only Texas Instruments TI-30X calculator.
- Give your answers in exact form, not decimals, unless instructed otherwise.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- **Check your work carefully.** We will award only limited partial credit.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 6 pages, plus this cover sheet. Make sure that your exam is complete.

Question	Points	Score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
Total	70	

1. (10 points) Estimate the area under the graph of the function $f(x) = \sqrt{x}(\ln x)^2$ between $x = 1$ and $x = 4$. Use a Riemann sum with 4 subintervals. Use the left point rule. Give your answer as a decimal number with 4 significant digits.

Solution:

We have $\Delta x = (4-1)/4 = 3/4 = 0.75$. The formula for the left point sum approximation L_n with $n = 4$ is

$$L_4 = 0.75(\sqrt{1}(\ln 1)^2 + \sqrt{1.75}(\ln 1.75)^2 + \sqrt{2.5}(\ln 2.5)^2 + \sqrt{3.25}(\ln 3.25)^2) \approx \boxed{3.18469}.$$

2. (10 points) Find the total (unsigned) area between the graphs of $f(x) = (2/\pi)^3 x^3$ and $g(x) = \sin x$. Give the answer in exact form.

   Hint: Sketch the graphs of the functions. Find x such that $f(x) = 1$.

Solution:

We have $f(\pi/2) = (2/\pi)^3(\pi/2)^3 = 1$ and $g(\pi/2) = \sin(\pi/2) = 1$. Similarly, $f(-\pi/2) = (2/\pi)^3(-\pi/2)^3 = -1$ and $g(-\pi/2) = \sin(-\pi/2) = -1$. Also, $f(0) = (2/\pi)^3 0^3 = 0$ and $g(0) = \sin 0 = 0$. We conclude that the graphs of the functions f and g intersect at $x = -\pi/2, 0$ and $\pi/2$.

A sketch of the graphs of the two functions shows that the graph of g lies below the graph of f on the interval $(-\pi/2, 0)$ and the graph of f lies below the graph of g on the interval $(0, \pi/2)$.

The area between the two graphs is equal to

$$\int_{-\pi/2}^{0} ((2/\pi)^3 x^3 - \sin x)dx + \int_{0}^{\pi/2} (\sin x - (2/\pi)^3 x^3)dx$$

$$= ((2/\pi)^3(1/4)x^4 + \cos x)\Big|_{-\pi/2}^{0} + (-\cos x - (2/\pi)^3(1/4)x^4)\Big|_{0}^{\pi/2}$$

$$= (1 - (2/\pi)^3(1/4)(-\pi/2)^4) + (-(2/\pi)^3(1/4)(\pi/2)^4 - (-1)) = \boxed{2 - \pi/4} \approx 1.2146$$

3. Compute the following integrals. In part (a), give the answer in the decimal form with at least three significant digits.

   (a) (10 points) $\displaystyle\int_2^3 \frac{1}{x^2}\tan\left(\frac{x+1}{x}\right)dx$

   Solution:
   We use the substitution rule with $u = \frac{x+1}{x}$, $du = -(1/x^2)dx$. When $x = 2$ then $u = 3/2$. When $x = 3$ then $u = 4/3$.

   $$\int_2^3 \frac{1}{x^2}\tan\left(\frac{x+1}{x}\right)dx = \int_{3/2}^{4/3}(-\tan(u))du = \int_{4/3}^{3/2}\tan(u)du = \ln|\sec u|\Big|_{u=4/3}^{u=3/2}$$

   $$= \ln|\sec(3/2)| - \ln|\sec(4/3)| \approx \boxed{1.2016}$$

   (b) (10 points) $\displaystyle\int \csc^2 z(\cot z + \sin^2 z)dz$

   Solution:
   Recall that $\csc z = 1/\sin z$. We have

   $$\int \csc^2 z(\cot z + \sin^2 z)dz = \int (\csc^2 z\cot z + (1/\sin z)^2\sin^2 z)dz = \int \csc z\cot z dz + \int 1dz.$$

   For the second integral, we have $\int 1dz = z + C$. For the first integral, we use the substitution $u = \cot z$, $du = -\csc^2 z dz$.

   $$\int \csc^2 z\cot z dz = \int -u du = -u^2/2 + C = -(\cot z)^2/2 + C$$

   $$\int \csc z(\cot z + \sin^2 z)dz = \boxed{-(\cot z)^2/2 + z + C}$$

4. (10 points) Find $f'(\pi/4)$ if
$$f(x) = \int_{\sin x}^{\cos x} \left(s^2 + s\right) ds.$$
Give your answer in exact simplified form.

Solution:

$$f'(x) = (\cos^2 x + \cos x)(-\sin x) - (\sin^2 x + \sin x)\cos x$$

Recall that $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$.

$$f'(\pi/4) = (1/2 + \sqrt{2}/2)(-\sqrt{2}/2) - (1/2 + \sqrt{2}/2)\sqrt{2}/2 = \boxed{-1 - \sqrt{2}/2} \approx -1.70711$$

5. (10 points) An object was thrown up from the ground level. It reached the highest point on its trajectory after 3 seconds.

   (i) What was the initial upward velocity?

   (ii) How high above the ground was the highest point on the trajectory?

   Use the metric system. Assume that the gravitational acceleration is -9.8m/sec^2. Give your answers as decimal numbers with 3 significant digits.

Solution:

Let $s(t)$ be the altitude of the object, $v(t)$ its velocity, and $a(t)$ its acceleration. We know that $a(t) = -9.8$ so

$$v(t) = \int a(t)dt = \int (-9.8)dt = -9.8t + C_1.$$

Let the initial velocity be called v_0. Then $v(0) = v_0$ and

$$v(0) = -9.8 \cdot 0 + C_1 = v_0$$

so $C_1 = v_0$.

$$v(t) = -9.8t + v_0.$$

We have

$$s(t) = \int v(t)dt = \int (-9.8t + v_0)dt = -4.9t^2 + v_0 t + C_2.$$

We know that $s(0) = 0$.

$$s(0) = -4.9 \cdot 0^2 + v_0 \cdot 0 + C_2 = 0$$

$$C_2 = 0$$

$$s(t) = -4.9t^2 + v_0 t$$

Let the maximum altitude be called m. Then there is only one t_m such that $s(t_m) = m$. Hence the equation

$$-4.9t^2 + v_0 t_m = m$$

has only one solution. The quadratic equation

$$-4.9t^2 + v_0 t - m = 0$$

has solutions

$$t_{1,2} = \frac{-v_0 \pm \sqrt{v_0^2 - 4(-4.9)(-m)}}{2(-4.9)}.$$

They reduce to a single solution if

$$v_0^2 - 4(-4.9)(-m) = 0 \qquad (*)$$

and then the only solution is $t_m = \frac{-v_0}{2(-4.9)}$. So $3 = \frac{-v_0}{2(-4.9)}$. Solving for v_0, we obtain $v_0 = 6 \cdot 4.9 = \boxed{29.4}$ (m/sec).

We now substitute $v_0 = 29.4$ into the equation (*) to see that

$$29.4^2 - 4(-4.9)(-m) = 0.$$

Solving for m yields $m = 29.4^2/(4 \cdot 4.9) = \boxed{44.1}$ (m).

6. (10 points) Consider the region A above the x-axis, below the graph of $y = \frac{1}{x}e^x \sin(e^x)$, between $x = \ln(\pi/2)$ and $x = \ln \pi$. Compute the volume of the solid of revolution obtained by rotating the region A about the y-axis. Do not compute the area of the region A. Give the answer in exact form.

Solution:

We will use the shell method. The volume is equal to

$$\int_{\ln(\pi/2)}^{\ln \pi} 2\pi x \left(\frac{1}{x}e^x \sin(e^x)\right) dx = \int_{\ln(\pi/2)}^{\ln \pi} 2\pi e^x \sin(e^x) dx.$$

We use the substitution $u = e^x$ with $du = e^x dx$. If $x = \ln(\pi/2)$ then $u = \pi/2$. If $x = \ln \pi$ then $u = \pi$.

$$\int_{\ln(\pi/2)}^{\ln \pi} 2\pi e^x \sin(e^x) dx = \int_{\pi/2}^{\pi} 2\pi \sin u du = -2\pi \cos u \Big|_{\pi/2}^{\pi} = \boxed{2\pi} \approx 6.283$$