Aut18 Mid1

1. (12 points) Evaluate the following integrals. Show all steps. Simplify and box your answer.

(a) $\int \frac{10x^2}{\sqrt{2-x^3}} + \frac{3}{\sqrt{1-x^2}}\,dx$ $= \underbrace{10\int \frac{x^2}{\sqrt{2-x^3}}\,dx}_{\text{u-sub}} + \underbrace{3\int \frac{1}{\sqrt{1-x^2}}\,dx}_{\text{Formula}}$

u-sub: $u = 2-x^3$

$-\frac{1}{3}du = \frac{\cancel{-3}x^2dx}{\cancel{-3}}$

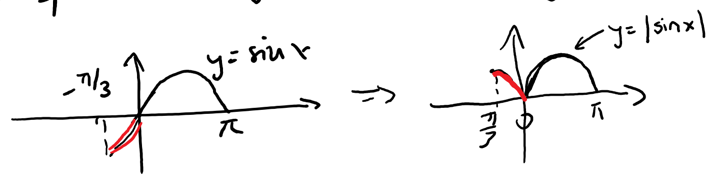
$= 10\int \frac{-1/3}{\sqrt{u}}\,du + 3\arcsin(x)$

$= -\frac{10}{3}\frac{u^{1/2}}{1/2} + 3\arcsin(x) + C$

$= \boxed{-\frac{20}{3}\sqrt{2-x^3} + 3\arcsin(x) + C}$

(b) $\int_{-\pi/3}^{\pi} |\sin(x)|\,dx$ We need to split this integral based on the sign of $\sin(x)$:

$$|\sin x| = \begin{cases} \sin x & \text{for } 0 \le x \le \pi \\ -\sin x & \text{for } -\frac{\pi}{3} \le x \le 0 \end{cases}$$



$$\int_{-\pi/3}^{\pi} |\sin x|\,dx = \int_{-\pi/3}^{0} -\sin x\,dx + \int_0^{\pi} \sin x\,dx$$

$$= \cos x\Big|_{-\pi/3}^{0} + (-\cos x)\Big|_0^{\pi}$$

$$= \cos(0) - \cos\left(-\frac{\pi}{3}\right) + (-\cos\pi + \cos 0)$$

$$= \left(1 - \frac{1}{2}\right) + (-(-1)+1)$$

$$= \frac{1}{2} + 2 = \boxed{\frac{5}{2}} = \boxed{2.5}$$

2. (6 points) A car travels along a straight road. The following table contains sample points of the velocity of the car, sampled every 10 minutes over the first hour of driving.

t (hrs.)	v(t) (mph)
0	20
1/6	40
2/6	-20
3/6	-40
4/6	-30
5/6	10
1	60

Use the **right endpoints** to estimate:

(a) the total distance driven in the first hour: $= \int_0^1 |v(t)| dt$

$$R_6 = \sum_{i=1}^{6} |v(t_i)| \underbrace{\Delta t}_{1/6} = (40 + 20 + 40 + 30 + 10 + 60)\frac{1}{6} = \boxed{\frac{200}{6}} = \boxed{\frac{100}{3}} \text{ miles.}$$

(b) the displacement of the car after the first hour: $\int_0^1 v(t) dt$

$$R_6 = \sum_{i=1}^{6} v(t_i) \Delta t = (40 - \cancel{20} - \cancel{40} - 30 + 10 + \cancel{60})\frac{1}{6} = \boxed{\frac{20}{6}} = \boxed{\frac{10}{3}} \text{ miles}$$

3. (4 points) Which of the functions labeled $F(x)$ in (a)-(d) below satisfy both conditions:

$$F'(x) = e^{x^2} \textbf{ and } F(2) = 0?$$

For each, state Yes or No. If "No" indicate which of the conditions fail.

(a) $F(x) = \int_0^x e^{t^2} dt$ $\boxed{\text{No}}$ $F(2) \neq 0$ [$F(2) = \int_0^2 e^{t^2} dt$ is a positive number ($\neq 0$) since it's the integral w/ increasing bounds of a positive fct. The other condition holds: $F'(x) = e^{x^2}$ (FTC I)]

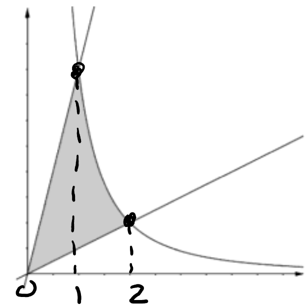
(b) $F(x) = \int_2^x e^{t^2} dt$ $\boxed{\text{Yes}}$ Both conditions hold: $F'(x) = e^{x^2}$ (FTC I) and $F(2) = \int_2^2 e^{t^2} dt = 0$

(c) $F(x) = \int_0^2 e^{t^2} dt$ $\boxed{\text{No}}$ Both fail: $F'(x) \neq 0$ ($\int_0^2 e^{t^2} dt$ is a <u>number</u> so its derivative $= 0 \neq e^{x^2}$)
$F(2) \neq 0$ (same reason as (a))

(d) $F(x) = \int_4^{x^2} e^t dt$ $\boxed{\text{No}}$ $F'(x) = e^{x^2} \cdot (2x) \neq e^{x^2}$ (FTC I + CHAIN rule)
(the other condition is true: $F(2) = \int_4^4 \cdots = 0$)

4. (8 points) Find the area of the region bounded by the curves

$$y = \frac{8}{x^2},\ y = 8x,\ \text{and } y = x.$$



Curves intersect:

$\frac{8}{x^2} = 8x \Rightarrow x^3 = 1 \Rightarrow \boxed{x=1}$

$\frac{8}{x^2} = x \Rightarrow x^3 = 8 \Rightarrow \boxed{x=2}$

$$\text{Area} = \int_0^1 (8x - x)\,dx + \int_1^2 \left(\frac{8}{x^2} - x\right) dx$$

$$= \int_0^1 7x\,dx + \int_1^2 8x^{-2} - x\,dx$$

$$= \frac{7}{2}x^2 \Big|_0^1 + \left(\frac{-8}{x} - \frac{1}{2}x^2\right)\Big|_1^2$$

$$= \frac{7}{2}(1^2 - 0^2) + \left[(-4-2) - \left(-8 - \frac{1}{2}\right)\right]$$

$$= \frac{7}{2} + \frac{5}{2}$$

$$= \boxed{6}$$

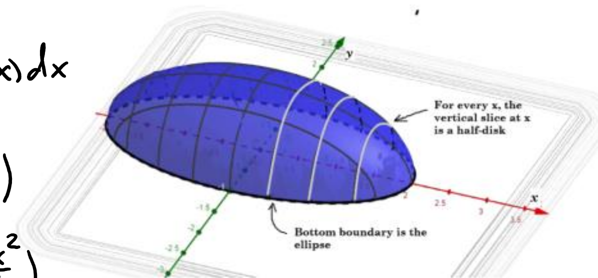
5. (6 points) Compute the volume of the solid pictured below. Its bottom side is bounded in the xy-plane by the ellipse:

$$x^2/4 + y^2 = 1.$$

All vertical slices through this solid that are perpendicular to the x-axis at x-values in the interval $-2 < x < 2$ are half-disks.

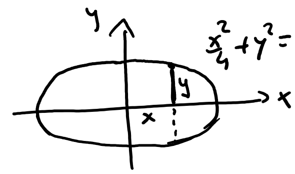
by symmetry

$$V = \int_{-2}^{2} A(x)\,dx = 2\int_{0}^{2} A(x)\,dx$$



$$A(x) = \frac{1}{2}\pi R^2(x) \quad \text{(half disk)}$$

$$= \frac{1}{2}\pi y^2 = \frac{1}{2}\pi\left(1 - \frac{x^2}{4}\right)$$



$$\frac{x^2}{4} + y^2 = 1 \Rightarrow y^2 = 1 - \frac{x^2}{4}.$$

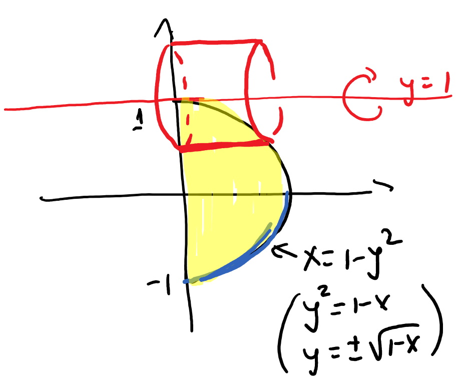
$$\therefore\ V = 2\int_0^2 \frac{1}{2}\pi\left(1 - \frac{x^2}{4}\right) dx$$

$$= \pi\left(x - \frac{1}{12}x^3\right)\Big|_0^2$$

$$= \pi\left(2 - \frac{1}{12}8\right) = \boxed{\frac{4}{3}\pi} \text{ (cubic units)}$$

6. (6 points) Sketch a picture of the region R entirely enclosed by the curve $x = 1 - y^2$ and the y-axis.

SET UP (but DO NOT COMPUTE) an integral equal to the volume of the solid of revolution obtained by rotating this region R around the **horizontal axis of rotation** $y = 1$.



Easiest way: integrate in y => horizontal rectangles generating SHELLS:

$$V = \int_{-1}^{1} 2\pi R(y) h(y)\,dy$$

$$\boxed{= \int_{-1}^{1} 2\pi (1-y)(1-y^2)\,dy}$$

Alternative method: Washers (if we integrate in x)

$$V = \int_0^1 \pi R^2(x) - \pi r^2(x)\,dx$$

$$= \boxed{\int_0^1 \pi\Big(1 - \underbrace{(-\sqrt{1-x})}_{\text{lower } \frac{1}{2} \text{ of parabola}}\Big)^2 - \pi\Big(1 - \underbrace{\sqrt{1-x}}_{\text{upper } \frac{1}{2} \text{ of parabola}}\Big)^2 dx.}$$

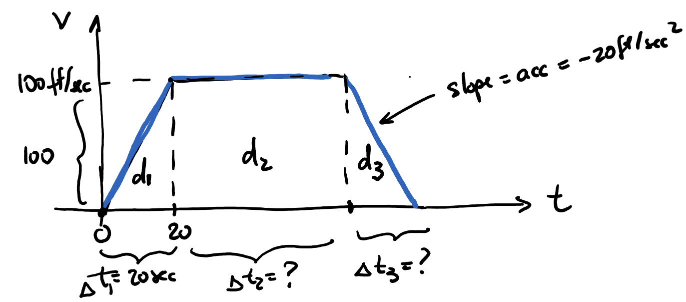
7. (8 points) A car drives along a straight road, from a point A to a point B, which is 3000 ft away from point A.

   The car starts at rest at point A, accelerates uniformly to its maximum speed of 100 ft/sec in 20 seconds, then drives at maximum speed for a while, before finally braking at a constant deceleration of 20 ft/sec^2 and coming to a complete stop at point B.

   How long does it take the car to complete this trip, from A to B?

There are 3 parts to this journey:
1) accelerating portion (20 sec.)
2) middle portion at 100 ft/sec constant speed
3) decelerating portion.

We need to compute time & distance for each of those 3 parts. We could do this in a number of different ways. Probably the easiest one is to graph the multi-part velocity function and to measure the distance for each portion as the AREA under the velocity graph:



1) Know $\Delta t_1 = 20$ sec and we can compute $d_1 = \frac{1}{2}(100)(20)$
$= 1000$ ft

[area of triangle, or $\int_0^{20} v(t)\,dt = \int_0^{20} (5t)\,dt$]

3) Know slope = acc = -20 ft/sec^2, from initial $v = 100$ ft/sec to final $v = 0$ ft/sec
Hence $\Delta t_3 = \frac{\Delta v}{-20} = \frac{-100}{-20} = 5$ sec.
Hence $d_3 = \frac{1}{2}(100)(5) = 250$ ft.
} The car takes 5 sec & 250 ft to come to a stop.

2) Which means that the car traveled $d_2 = 3000 - d_1 - d_3$
$= 3000 - 1000 - 250 = 1750$ ft
at max speed in the middle portion.
Hence it took $\Delta t_2 = \frac{1750 \text{ ft}}{100 \text{ ft/sec}} = 17.50$ sec.

Total time $= 20$ sec $+ 17.5$ sec $+ 5$ sec $= \boxed{42.5 \text{ sec}}$