Solutions (v1)

1. (10 points) Evaluate the following integrals. Show your steps. Simplify and box your final answer.

(a) $\int \left(\sin(x)\cos(x) + \frac{3\sin(x)}{\cos^2(x)} - 2\right) dx$

$$= \underbrace{\int \left(\cos x + \frac{3}{\cos^2 x}\right) \sin x \, dx}_{\boxed{u = \cos x \quad du = -\sin x \, dx}} - \underbrace{\int 2 \, dx}$$

$$= \int \left(u + \frac{3}{u^2}\right)(-1)\, du \qquad - 2x \qquad + C$$

$$= -\int u \, du - 3\int u^{-2} du - 2x + C$$

$$= -\frac{1}{2}u^2 + \frac{3}{u} - 2x + C = \boxed{-\frac{1}{2}\cos^2 x + \frac{3}{\cos x} - 2x + C}$$

OR

Using identities: $\sin x \cos x = \frac{1}{2}\sin(2x)$, $\frac{\sin x}{\cos x} = \tan x$, $\frac{1}{\cos x} = \sec x$.

$$\int \sin x \cos x \, dx + 3\int \frac{\sin x}{\cos x}\frac{1}{\cos x} dx - \int 2 \, dx$$

$$= \int \frac{\sin(2x)}{2} dx + 3\int \tan x \sec x \, dx - 2x + C = \boxed{\frac{-\cos(2x)}{4} + 3\sec x - 2x + C}$$

(b) $\int_3^4 \frac{x^2}{(x-2)^2} dx$ $\boxed{\begin{array}{l} u = x-2 \\ du = dx \end{array}} \Leftrightarrow \boxed{x = u+2}$

$$= \int_{u=3-2}^{u=4-2} \frac{(u+2)^2}{u^2} du$$

$$= \int_1^2 \frac{u^2+4u+4}{u^2} du$$

$$= \int_1^2 1 + \frac{4}{u} + \frac{4}{u^2} du$$

$$= \left[u + 4\ln|u| - \frac{4}{u}\right]\Big|_1^2$$

$$= [\cancel{2} + 4\ln 2 - \cancel{2}] - [1 + 4\underbrace{\ln(1)}_{0} - 4]$$

$$= \boxed{4\ln 2 + 3}$$

2. A particle is moving in a straight line with an acceleration at t seconds of:

$$a(t) = 6t \text{ ft/s}^2.$$

At $t = 3$ seconds, the particle's velocity of the particle is measured to be $v(3) = 15$ ft/s.

(a) (4 points) Find the particle's velocity function, $v(t)$.

$$v(t) = \int a(t)\,dt = \int 6t\,dt = 3t^2 + C$$

since $v(3) = 15$: $\quad 3(3)^2 + C = 15 \Rightarrow C = 15 - 27 = -12$

$$\therefore \boxed{v(t) = 3t^2 - 12} \text{ ft/sec}$$

(b) (6 points) Find the **total distance** traveled by the particle in the first 3 seconds.

$$\begin{aligned}
\text{Total distance} &= \int_0^3 |v(t)|\,dt \\
&= \int_0^3 |3t^2 - 12|\,dt \\
&= \int_0^2 (12 - 3t^2)\,dt + \int_2^3 (3t^2 - 12)\,dt \\
&= \left[12t - t^3\right]\Big|_0^2 + \left[t^3 - 12t\right]\Big|_2^3 \\
&= \underbrace{\left[(24 - 8) - 0\right]}_{} + \underbrace{\left[(27 - 36) - (8 - 24)\right]}_{} \\
&= 16 + 7 \\
&= \boxed{23} \text{ feet}
\end{aligned}$$

Since $v(t)=0$ at $t=\pm 2$ and, on $[0,3]$ it changes sign at $t=2$

$-9+16$

3. (10 points) Consider the region bounded by:

$$y = \frac{1}{\sqrt{x}}, \text{ the } x\text{-axis, and the lines } x = 1 \text{ and } x = 16$$

(a) Compute the area A of this region. Show your work.

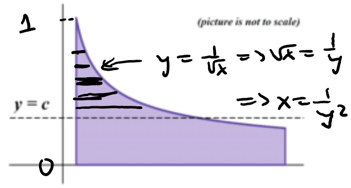
$$A = \int_1^{16} \frac{1}{\sqrt{x}}\,dx$$

$$= 2\sqrt{x}\Big|_1^{16}$$

$$= 8 - 2$$

$$= \boxed{6}$$

(b) Suppose the portion of the region that lies **above** the **horizontal line** $y = c$ is **one third of the total area** A of the region. Compute the value of c.



area above $y = c$:

$$\int_c^1 \left(\frac{1}{y^2} - 1\right) dy = \frac{1}{3}(\text{total area})$$

$$\left(-\frac{1}{y} - y\right)\Big|_c^1 = \frac{1}{3}(6) = 2$$

$$(-2) + \frac{1}{c} + c = 2$$

$$-2c + 1 + c^2 = 2c$$

$$c^2 - 4c + 1 = 0$$

$$c = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

Since $0 < c < 1 \Rightarrow \boxed{c = 2 - \sqrt{3}}$

4. (10 points) Let $\mathcal{R}$ denote the region bounded by the graphs of:

$$y = \ln(x),\ y = \ln(5),\ \text{the } x\text{-axis, and the } y\text{-axis}$$

(a) Compute the volume of the solid of revolution obtained by rotating this region $\mathcal{A}$ **about the y-axis**.

Disks, in y:

$$V_1 = \int_0^{\ln 5} \pi R^2\, dy$$

$$= \int_0^{\ln 5} \pi (e^y)^2\, dy$$

$$= \pi \int_0^{\ln 5} e^{2y}\, dy$$

$$= \frac{1}{2}\pi e^{2y}\Big|_0^{\ln 5}$$

$$= \frac{1}{2}\pi\left(e^{2\ln 5} - e^0\right)$$

$$= \frac{1}{2}\pi(25-1)$$

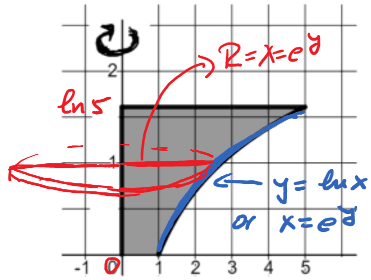
$$= \boxed{12\pi}.$$

OR $\boxed{u = e^y,\ du = e^y\,dy}$

$$= \int_{u=e^0=1}^{u=e^{\ln 5}=5} \pi u\, du$$

$$= \int_1^5 \pi u\, du = \frac{1}{2}\pi u^2\Big|_1^5$$

$$= \frac{1}{2}\pi(25-1) = \boxed{12\pi}$$

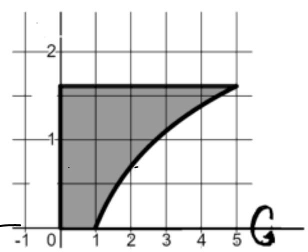


(b) Express the volume of the solid of revolution obtained by rotating this region $\mathcal{R}$ **around the x-axis** as an integral or as a sum/difference of integrals.
Do not evaluate the integral(s), just write down the expression.

In x: Disks on [0,1] + Washers on [1,5]

$$V_2 = \int_0^1 \pi R^2\, dx + \int_1^5 \pi R^2 - \pi r^2\, dx$$

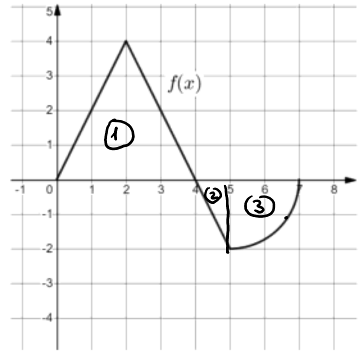
$$= \boxed{\int_0^1 \pi(\ln 5)^2\, dx + \int_1^5 \pi(\ln 5)^2 - \pi(\ln x)^2\, dx}$$



OR In y: Shells on $[0, \ln 5]$

$$V_2 = \int_0^{\ln 5} 2\pi R h\, dy = \boxed{\int_0^{\ln 5} 2\pi\, y\, e^y\, dy}$$

5. (10 points) The figure on the right shows a function $y = f(x)$ whose graph consists of two line segments and a quarter of a circle. Use this graph to find each of the following quantities.



Leave your answers in exact form. SHOW WORK.

(a) $\int_0^7 f(x)\,dx =$ area 1 − area 2 − area 3

$= \frac{1}{2}(4)(4) - \frac{1}{2}(2)(1) - \frac{\pi(2)^2}{4}$

$= 8 - 1 - \pi$

$= \boxed{7-\pi}$

(b) Define a function $F(x) = \int_0^x f(t)\,dt$.
What is the maximal value of $F(x)$ in the interval $[0,7]$, and at what value of x is it reached?

$F(0)=0$, $F(x) \nearrow$ on $[0,4]$ to $F(4)=8$, then $\searrow$ on $[4,7]$ to

$\boxed{\text{Max value of } F(x) \text{ is } F(4)=8 \text{ at } x=4}$

(c) Define a function $G(x) = \int_{x^2}^0 f(t)\,dt$. Compute $G'(1)$.

$G'(x) = -\frac{d}{dx}\int_0^{x^2} f(t)\,dt = -f(x^2)\cdot 2x$ (FTC I + Chain Rule)

$\therefore G'(1) = -f(1)\cdot 2$ (On graph: $f(1) = 2$)

$\therefore \boxed{G'(1) = -4}$

(d) Compute $\lim_{n\to\infty}\sum_{i=1}^{n}[1+f(2i/n)]\cdot(2/n) = \lim_{n\to\infty}\sum_{i=1}^{n}(1+f(x_i))\,\Delta x = \int_0^2 (1+f(x))\,dx$

$\left[\begin{array}{l} \Delta x = \frac{2}{n} \Rightarrow [a,b]=[0,2] \\ x_i = \frac{2i}{n} \Rightarrow f(\frac{2i}{n}) = f(x_i) \end{array}\right]$

$\int_0^2 (1+f(x))\,dx = \underbrace{\int_0^2 1\,dx}_{2} + \underbrace{\int_0^2 f(x)\,dx}_{\frac{1}{2}(2)(4)}$

$= 2 + \frac{1}{2}(2)(4)$

$= \boxed{6}$