University of Washington **Department of Mathematics**
Midterm Exam #1 - Solutions **Math 125E** **January 29, 2003**

1. (a) (4 points) We use the substitution $u = 1 - \sin x$ so that $du = -\cos x\,dx$. The corresponding limits at $x = 0$ and $x = \pi/2$ become $u = 1$ and $u = 0$ respectively. We then have

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx = -\int_1^0 u^{-1/2}\,du = 2u^{1/2}|_0^1 = 2.$$

(b) (4 points) We use the substitution $u = e^{x^2} - 5$ so that $du = 2x\,e^{x^2}\,dx$. This leads to

$$\int \frac{5xe^{x^2}}{e^{x^2}-5}\,dx = \frac{5}{2}\int \frac{2xe^{x^2}}{e^{x^2}-5}\,dx = \frac{5}{2}\int \frac{1}{u}\,du = \frac{5}{2}\ln|u| + C = \frac{5}{2}\ln|e^{x^2}-5| + C.$$

(c) (4 points) We use the substitution $u = \ln(\cos x)$ then $du = \frac{-\sin x}{\cos x}\,dx = -\tan x\,dx$. This leads to

$$\int \tan x \ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C.$$

2. (8 points) Find the derivative of the function $g(x) = \int_1^{\cos x} \sqrt[3]{1-t^2}\,dt$.
Let $y = g(x)$ and $u = \cos x$. Then by the chain rule and the Fundamental Theorem of calculus, part 1, we have

$$g'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \sqrt[3]{1-u^2}(-\sin x) = \sqrt[3]{1-\cos^2 x}(-\sin x) = -\sin x\sqrt[3]{\sin^2 x} = -(\sin x)^{5/3}.$$

3. (8 points) A particle moves along a line with velocity function $v(t) = t^2 - t$.

(a) (4 points) Find the displacement of the particle during the time interval $[0, 5]$.

$$\text{displacement} = \int_0^5 (t^2 - t)\,dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_0^5 = \frac{125}{3} - \frac{25}{2} = \frac{175}{6} = 29.1\overline{6}.$$

(a) (4 points) Find the distance traveled by the particle during the time interval $[0, 5]$.

$$\begin{aligned} \text{distance traveled} &= \int_0^5 |t^2 - t|\,dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 (t - t^2)\,dt + \int_1^5 (t^2 - t)\,dt \\ &= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 + \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_1^5 = \frac{1}{2} - \frac{1}{3} - 0 + \left(\frac{125}{3} - \frac{25}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) \\ &= \frac{177}{6} = 29.5. \end{aligned}$$

4. (8 points) Find a number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

By the symmetry of the region about the $y - axis$, we may consider on the area in the first quadrant where $y = x^2 \Longrightarrow x = \sqrt{y}$. We are looking for a number b such that

$$\int_0^4 \sqrt{y}\, dy = 2\int_0^b \sqrt{y}\, dy$$

(the integral on the left is, by the symmetry, equal to half the area of the region). This implies

$$\frac{2}{3}\left[y^{3/2}\right]_0^4 = \frac{4}{3}\left[y^{3/2}\right]_0^b \quad \Longrightarrow \quad \frac{2}{3}(8-0) = \frac{4}{3}(b^{3/2}-0) \quad \Longrightarrow \quad b^{3/2} = 4 \quad \Longrightarrow \quad b = 4^{2/3}.$$

5. (14 points) Consider the region R bounded by the curves $y = \frac{1}{2}\sqrt{x}$, $y = 1$ and the $y -$ axis.
(a) (5 points) Find the area of R.

$$A = \int_0^4 (1 - \frac{1}{2}\sqrt{x})\, dx = \left[x - \frac{1}{3}x^{3/2}\right]_0^4 = \frac{4}{3}.$$

(b) (5 points) Find the volume, V, of the solid obtained by rotating R about the x-axis.
Slicing perpendicular to the x-axis the cross-sections are washers with area $A(x) = \pi R(x)^2 - \pi r(x)^2 = \pi(1 - \frac{x}{4})$. The volume is obtained by integrating these areas from $x = 0$ to $x = 4$ giving

$$V = \int_0^4 \pi(1 - \frac{x}{4})\, dx = \pi(x - \frac{x^2}{8})|_0^4 = \pi(4-2) = 2\pi.$$

Alternatively we can solve for x as a function of y to obtain $x = 4y^2$ and the use the method of cylindrical shells. For $0 \leq y \leq 1$, the radius of the shell is y and the height is $4y^2$. So the volume of the solid is given by

$$V = \int_0^1 2\pi y 4y^2\, dy = 8\pi \int_0^1 y^3\, dy = 8\pi \frac{y^4}{4}|_0^1 = 2\pi.$$

(c) (4 points) Set up an integral for the volume of the solid obtained by rotating the region R about the line $x = 5$. DO NOT EVALUATE THE INTEGRAL.
Slicing perpendicular to the line $x = 5$ the cross-sections are washers with area $A(y) = \pi R(x)^2 - \pi r(x)^2 = \pi(5^2 - (5-4y^2)^2)$. The volume is therefore given by

$$V = \int_0^1 \pi(5^2 - (5-4y^2)^2)\, dy.$$

Alternatively we can use the method of cylindrical shells. For $0 \leq x \leq 4$, the radius of the corresponding shell is $5 - x$, and the height of the shell is $1 - \frac{1}{2}\sqrt{x}$. The volume is therefore given by

$$V = \int_0^4 2\pi(5-x)(1 - \frac{1}{2}\sqrt{x})\, dx.$$

(Either of these yeilds a volume of $V = \frac{152}{15}\pi$.)