Math 125C Midterm 1 Solutions Autumn 2003

1.(a) $\displaystyle\int_1^{\sqrt{3}} \frac{5}{1+y^2}\,dy = 5\tan^{-1} y\Big|_1^{\sqrt{3}} = 5\pi/3 - 5\pi/4 = 5\pi/12$

(b) Let $u = \tan\theta$ so $du = \sec^2\theta\,d\theta$. Then $\displaystyle\int_0^{\pi/4} \sec^2\theta\,\cos\Big(\tan\theta\Big)\,d\theta = \int_0^1 \cos u\,du = \sin u\Big|_0^1 = \sin(1)$

(c) Let $v = 2-x$ so that $dv = -dx$ and $3x^2 = 3(2-v)^2 = 12 - 12\,v + 3\,v^2$. Then

$$\begin{aligned}
\int \frac{3x^2}{\sqrt{2-x}}\,dx &= -\int \frac{12-12\,v+3\,v^2}{\sqrt{v}}\,dv \\
&= \int -12\,v^{-1/2} + 12\,v^{1/2} - 3\,v^{3/2}\,dv \\
&= -24\,v^{1/2} + 8\,v^{3/2} - 6/5\,v^{5/2} + C \\
&= -24\sqrt{2-x} + 8\,(2-x)^{3/2} - 6/5\,(2-x)^{5/2} + C
\end{aligned}$$

2. Note that $4t - t^3 \geq 0$ when $0 \leq t \leq 2$, and that it is negative on the rest of the interval. Thus

$$\begin{aligned}
\int_{-1}^3 \left|4t - t^3\right|\,dt &= -\int_{-1}^0 4t - t^3\,dt + \int_0^2 4t - t^3\,dt - \int_2^3 4t - t^3\,dt \\
&= -\left(2t^2 - 1/4\,t^4\Big|_{-1}^0\right) + \left(2t^2 - 1/4\,t^4\Big|_0^2\right) - \left(2t^2 - 1/4\,t^4\Big|_2^3\right) \\
&= 7/4 + 4 + 25/4 = 12
\end{aligned}$$

3. Write $f(x) = -\displaystyle\int_9^{x^2} \cos\Big(\pi\sqrt{t}\Big)\,dt$. By the Fundamental Theorem of Calculus, $f'(x) = -2x\,\cos\Big(\pi\sqrt{x^2}\Big)$.
Now $f(3) = -\displaystyle\int_9^9 \cos\Big(\pi\sqrt{t}\Big)\,dt = 0$ and $f'(3) = -6\,\cos(3\pi) = 6$. Thus the line is $y - 0 = 6(x-3)$.

4. The equations of the lines are $y = x+1$, $y = -2(x-5)$, and $y + 4 = \frac{6}{5}(x-7)$.

(a) $f(4) = -2(4-5) = 2 \qquad f(8) = \frac{6}{5}(8-7) - 4 = -14/5$

(b) To compute $g(5)$, divide the area at $x = 3$. The left part is a trapeziod with base $b = 3$ and heights $h_1 = 1$, $h_2 = 4$. The area of this part is $\frac{1}{2}\,b\,(h_1 + h_2) = 15/2$. The right part is a triangle with base $b = 2$ and height $h = 4$. The area of this part is $\frac{1}{2}\,b\,h = 4$. Thus $g(5) = 15/2 + 4 = 23/2$.
To compute $g(9)$, note that it is $g(5)$ minus the area under the x-axis from $x = 5$ to $x = 9$. Split this area at $x = 7$ to get a triangle of area 4 and a trapezoid with base $b = 2$ and heights $h_1 = 4$, $h_2 = |f(9)| = 8/5$. Thus the area under the x-axis is $4 + \frac{1}{2}\,(2)\,(4 + \frac{8}{5}) = 48/5$ and $g(9) = 23/2 - 48/5 = 19/10$.

(c) Since $g'(x) = f(x)$ the function $y = g(x)$ has only one relative maximum, at $x = 5$. Thus the max must occur at $x = 5$ or at one of the endpoints $x = 0$ and $x = 12$. But $g(0) = 0$, $g(5) = 23/2$ and we can compute that $g(12) = 5/2$. Thus the max is $g(5) = 23/2$.

5. For both parts you need to solve $x^2 = x^3 - x^2$. This gives $0 = x^3 - 2x^2 = x^2\,(x-2)$ so the limits of integration in both problems are $x = 0$ to 2.

(a) $\pi\displaystyle\int_0^2 \Big(4 - x^3 + x^2\Big)^2 - \Big(4 - x^2\Big)^2\,dx$

(b) $2\pi\displaystyle\int_0^2 x\,\Big(2x^2 - x^3\Big)\,dx$