1 (12 points) Compute the following integrals. Give your answers in exact form.

(a) (4 points) $\displaystyle\int_1^4 \frac{\sqrt{z}+3}{z}\,dz$

$$\begin{aligned}\int_1^4 \frac{\sqrt{z}+3}{z}\,dz &= \int_1^4 z^{-1/2}+3z^{-1}\,dz \\ &= 2z^{1/2}+3\ln|z|\,\Big|_1^4 \\ &= 2+3\ln 4\end{aligned}$$

(b) (4 points) $\displaystyle\int_0^{\pi/4} \frac{\sin t}{\cos^3 t}\,dt$

Set $u=\cos t$ *and* $du=-\sin t\,dt$.
Note that $\cos(0)=1$ *and* $\cos(\pi/4)=\sqrt{2}/2$.

$$\begin{aligned}\int_0^{\pi/4} \frac{\sin t}{\cos^3 t}\,dt &= \int_1^{\sqrt{2}/2} -u^{-3}\,du \\ &= \int_{\sqrt{2}/2}^1 u^{-3}\,du \quad \textit{just for fun!} \\ &= -\frac{1}{2}u^{-2}\,\Big|_{\sqrt{2}/2}^1 \\ &= \frac{1}{2}\end{aligned}$$

(c) (4 points) $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

Set $v=x^3$ *and* $dv=3x^2\,dx$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^6}}\,dx &= \frac{1}{3}\int \frac{dv}{\sqrt{1-v^2}}\,dv \\ &= \frac{1}{3}\sin^{-1} v + C \\ &= \frac{1}{3}\sin^{-1} x^3 + C\end{aligned}$$

2 (8 points) Let $f(x) = \int_1^{2x-1} 3t^2 + \ln(t)\, dt$. Find the equation of the tangent line to $y = f(x)$ at $x = 1$.

We use the point-slope equation $y - b = m(x - a)$ *with* $a = 1$.

Then $b = f(1) = \int_1^1 3t^2 + \ln(t)\, dt = 0$.

To compute m*, first calculate* $f'(x) = \left[3(2x-1)^2 + \ln(2x-1)\right] \cdot 2$.

Then $m = f'(1) = 3 \cdot 2 = 6$.

The equation of the tangent line is $y = 6(x - 1)$.

3 (8 points) Calculate the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{999}{n} \cdot \frac{1}{1 + \frac{999}{n} i}$ by writing it as a definite integral and solving the integral.

Consider $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x = \int_a^b f(x)\, dx$.

Looks like $\Delta x = \frac{b-a}{n} = \frac{999}{n}$ *so* $b - a = 999$.

Then $x_i = a + i\Delta x = 1 + \frac{999}{n} i$ *so* $a = 1$. *Thus* $b = 1000$ *and* $f(x) = \frac{1}{x}$.

Calculate $\int_1^{1000} \frac{1}{x}\, dx = \ln|x| \Big|_1^{1000} = \ln(1000)$.

4 (12 points) Tafu is driving his car along a stright street. The velocity of his car is given by $v(t) = 90t^2 - 50t$ mi/hr, where t is measured in hours. Tafu reaches his destination after one hour. The car can drive 35 miles per gallon of fuel. How much fuel did Tafu use up for this journey?

We need to calculate the total miles and then divide by 35 mi/gal.

Total miles $= \int_0^1 |v(t)|\,dt$

Note that $0 = v(t) = 10t \cdot (9t - 5)$ *gives* $t = 0, 5/9.$

Check also that $v(t) < 0$ *for* $0 < t < 5/9$ *and* $v(t) > 0$ *otherwise.*

$$\begin{aligned}
\int_0^1 |v(t)|\,dt &= -\int_0^{5/9} 90t^2 - 50t\,dt + \int_{5/9}^1 90t^2 - 50t\,dt \\
&= -\left(30t^3 - 25t^2\right)\Big|_0^{5/9} + \left(30t^3 - 25t^2\right)\Big|_{5/9}^1 \\
&= \frac{625}{243} + \frac{1840}{243} \\
&= \frac{2465}{243} \approx 10.144\,\textit{miles}
\end{aligned}$$

Finally, $\dfrac{2465}{243} \div 35 = \dfrac{493}{1701} \approx 0.29$ *gallons. (About a third of a gallon.)*

5 (10 points) Let R be the region in the first quadrant bounded by $y = 6x - x^2$ and $y = x^3$. Set up an integral that computes the volume of the solid generated by rotating R around the line $y = -3$. DO NOT EVALUATE.

We'll use the method of washers $\pi \int_a^b r_2^2 - r_1^2 \, dx$. *We integrate with respect to* x *because the axis of rotation is horizontal.*

To find a *and* b*, solve the equation* $x^3 = 6x - x^2$.

$$0 = x^3 + x^2 - 6x = x(x-2)(x+3)$$

Since the region is in the first quadrant, we ue the roots $x = 0, 2$.

We can figure out which curve is higher in the interval $[0, 2]$ *by plugging in* $x = 1$.

$$6 \cdot 1 - 1^2 > 1^3$$

Thus $r_2 = (6x - x^2) - (-3)$ *and* $r_1 = x^3 - (-3)$.

Now we can compute the volume:

$$\pi \int_a^b r_2^2 - r_1^2 \, dx = \pi \int_0^2 (3 + 6x - x^2)^2 - (3 + x^3)^2 \, dx$$

(If you wanted to compute this, you would square everything out to get

$$\pi \int_0^2 36\,x + 30\,x^2 - 18\,x^3 + x^4 - x^6 \, dx$$

but this is not required on this problem.)