Math 125

Midterm 1 (January 30, 2020)

NAME: Solutions **Section:** ____________

- Time: you have **75 minutes**.
- Please show all work and justify your answers. The final answers must be "reasonably" simplified. For example, a rational number must be given in the form $\frac{a}{b}$ for some integers a and b, but it is ok to have expressions like $\ln 3$ or e^4 in your final answer.
- You are allowed to use calculator (Model TI-30X IIS only) and one *handwritten* (with your own handwriting) 8.5 x 11 inch sheet of notes. Writing allowed on both sides.
- Have your *Husky Card* visible on the desk beside you.
- You may use both sides of the paper.
- Make sure you have **9 pages** and **6 problems** before starting the exam.

Academic integrity is expected of all students at all times. Understanding this, I declare I shall not give, use, or receive unauthorized aid.

SIGNATURE: ______________________

Problem 1: ___ / 20
Problem 2: ___ / 20
Problem 3: ___ / 20
Problem 4: ___ / 20
Problem 5: ___ / 20
Problem 6: ___ / 20

Total: ___ / 120

Problem 1: Evaluate the following integrals:

(a)

$$\int_{-1}^{1} |x^2 - x| \, dx$$

(b)

$$\int \sqrt{\sqrt{t} + 1} \, dt$$

a) $|x^2 - x| = |x(x-1)| = \begin{cases} x(x-1) & -1 \leq x \leq 0 \\ -x(x-1) & 0 \leq x \leq 1 \end{cases}$

$$\int_{-1}^{1} |x^2 - x| dx = \int_{-1}^{0} (x^2 - x) dx - \int_{0}^{1} (x^2 - x) dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{0}^{1}$$

$$= \left(\frac{5}{6}\right) - \left(\frac{-1}{6}\right) = \boxed{1}$$

b) $$\int \sqrt{t^{1/2} + 1} \, dt = \int \sqrt{u} \, 2(u-1) du = 2\int (u^{3/2} - u^{1/2}) du = \frac{4}{5} u^{5/2} - \frac{4}{3} u^{3/2} + C$$

$$\left\{\begin{array}{l} u = t^{1/2} + 1 \\ du = \frac{1}{2} t^{-1/2} dt \rightarrow dt = 2t^{1/2} du = 2(u-1) du \end{array}\right\}$$

$$= \frac{4}{15} u^{3/2} (3u - 5) + C = \boxed{\frac{4}{15} (1 + \sqrt{t})^{3/2} (3\sqrt{t} - 2) + C}$$

Problem 2: Find the function $y = f(t)$ satisfying

$$y'' = t + \cos(t) \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0\,.$$

$$y' = \int (t + \cos(t))\,dt = \frac{t^2}{2} + \sin(t) + C$$

$$y = \int \left(\frac{t^2}{2} + \sin(t) + C\right)dt = \frac{t^3}{6} - \cos(t) + Ct + D$$

$$\begin{cases} 1 = y(0) = -1 + D \\ 0 = y'(0) = C \end{cases} \rightsquigarrow \begin{cases} C = 0 \\ D = 2 \end{cases}$$

So $\boxed{y = \frac{t^3}{6} - \cos(t) + 2}$

Problem 3: Consider the function

$$f(x) = \int_2^{x^2} \sqrt{1 + \ln\left(\frac{t}{2}\right)}\, dt\ .$$

(a) Evaluate $f'(x)$. Remember to show all work and justify your answer.

(b) Compute $f(\sqrt{2})$ and $f'(\sqrt{2})$.

a) Let $A(x) = \int_2^x \sqrt{1+\ln(\frac{t}{2})}\, dt$. By the Fundamental Theorem of calculus

we have $A'(x) = \sqrt{1+\ln(\frac{x}{2})}$

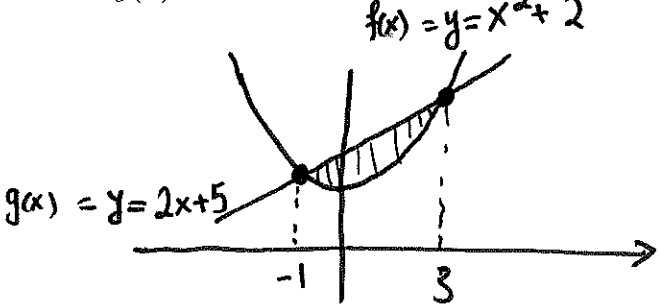
But $f(x) = A(x^2)$. By chain rule, $f'(x) = 2x A'(x^2)$. So

$$\boxed{f'(x) = 2x\sqrt{1+\ln(\frac{x^2}{2})}}$$

b) $f(\sqrt{2}) = \int_2^2 \cdots = \boxed{0}$

$f'(\sqrt{2}) = 2(\sqrt{2})\sqrt{1+\underbrace{\ln(\frac{2}{2})}_{\text{zero}}} = \boxed{2\sqrt{2}}$

Problem 4: Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$.



$$f(x) = g(x)$$

$$x^2 + 2 = 2x + 5 \rightsquigarrow x^2 - 2x - 3 = 0$$

$$x = -1 \text{ or } x = 3$$

$$\text{Area} = \int_{-1}^{3} (g(x) - f(x))\,dx = \int_{-1}^{3} (-x^2 + 2x + 3)\,dx = \left[-\frac{x^3}{3} + x^2 + 3x \right]_{-1}^{3}$$

$$= \frac{32}{3}$$

Problem 5: Evaluate the following limit:

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}$$

Hint: Use the theory of *Riemann sums* and express the limit as a *definite integral.*

consider $f(x)=\sqrt{x}$ on $[0,1]$

The Riemann Sum (with equal partitions and right endpoints as "samples")

is

$$R_n = \sum_{i=1}^{n} \Delta x\, f(0+i\Delta x) = \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$$

Since $f(x)=\sqrt{x}$ is continuous, it is integrable on $[0,1]$.

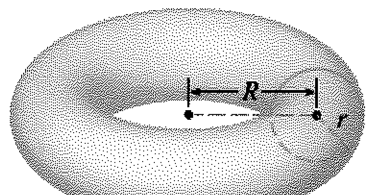
So we have

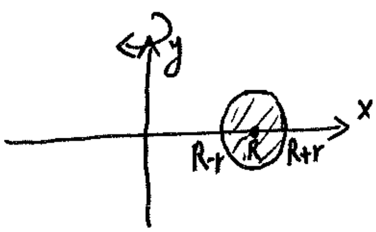
$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}} = \lim_{n\to+\infty} R_n = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

Problem 6: The *torus* (doughnut-shaped solid) in the figure is obtained by rotating the circle $(x-R)^2+y^2=r^2$ around the y-axis (assume $R>r$).

(a) Set up an integral for the volume of this torus.

(b) Find the volume of the torus by evaluating the integral.





$$(x-R)^2+y^2=r^2 \quad \begin{cases} \to y=\pm\sqrt{r^2-(x-R)^2} & \begin{cases} f_1(x)=\sqrt{r^2-(x-R)^2} \\ f_2(x)=-\sqrt{r^2-(x-R)^2} \end{cases} \\ \to x=R\pm\sqrt{r^2-y^2} & \begin{cases} g_1(y)=R+\sqrt{r^2-y^2} \\ g_2(y)=R-\sqrt{r^2-y^2} \end{cases} \end{cases}$$

Method 1 (Shells):

$$\text{Volume} = \int_{R-r}^{R+r} 2\pi x\left(f_1(x)-f_2(x)\right)dx = \int_{R-r}^{R+r} 2\pi x\left(2\sqrt{r^2-(x-R)^2}\right)dx = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du$$

$$\begin{cases} x-R=u \\ dx=du \end{cases}$$

$$= 4\pi\underbrace{\int_{-r}^{r} u\sqrt{r^2-u^2}\,du}_{\text{odd function}} + 4\pi R\underbrace{\int_{-r}^{r}\sqrt{r^2-u^2}\,du}_{\text{area of half-circle}}$$

$$= 0 + 4\pi R\left(\frac{\pi r^2}{2}\right) = \boxed{2\pi^2 R r^2}$$

Method 2 ("Washer")

$$\text{Volume} = \int_{-r}^{r}\pi(g_1(y))^2dy - \int_{-r}^{r}\pi(g_2(y))^2dy = \pi\int_{-r}^{r}4R\sqrt{r^2-y^2}\,dy$$

$$= 4\pi R\underbrace{\int_{-r}^{r}\sqrt{r^2-y^2}\,dy}_{\text{area of half-circle}} = 4\pi R\,\frac{\pi r^2}{2} = \boxed{2\pi^2 R r^2}$$