Name________________________________

Math 125 **Second Midterm** **11:30 a.m., Nov. 16, 2017**

(80 minutes — 100 points)

Please show all your work clearly, and cross out any erroneous work that you do not want considered. If you need more space, you can use the reverse side. A sheet of notes is permitted, but no calculator or other electronic device.

1. Find the following indefinite integrals:

(a) (20 points) $\displaystyle\int \frac{\cos(1/\sqrt{x})dx}{x^2}$

(b) (20 points) $\displaystyle\int \frac{x\,dx}{(2x-x^2)^{3/2}}$

(CONTINUED ON NEXT PAGE)

2. (20 points) Because $x^3 - 1$ has 1 as a root, it is the product of $x - 1$ and a quadratic factor (which cannot be factored). Using this, find the quadratic factor and then find the partial fraction decomposition of

$$\frac{7x^2 + 8}{x^3 - 1}.$$

Do **not** integrate it.

(CONTINUED ON NEXT PAGE)

3. (20 points) Determine whether or not the following improper integral converges. If not, explain how you know. If it does converge, determine its value exactly (as a multiple of π).

$$\int_0^\infty \frac{e^{x/2}dx}{1+e^x}.$$

(CONTINUED ON NEXT PAGE)

4. (20 points) Let R be the region above the line $y = 2$, below the line $y = 5$, to the right of the line $x = -1$, and to the left of the function $x = f(y)$. Suppose that you do not have a formula for $f(y)$, but only know it from a table that lists the seven values corresponding to equally-spaced numbers from 2 to 5: $f(2) = x_0$, $f(2.5) = x_1$, $f(3) = x_2$, $f(3.5) = x_3$, $f(4) = x_4$, $f(4.5) = x_5$, $f(5) = x_6$. The region R is rotated around the line $x = -1$ to form a container that's full of water. Set up the integral and use Simpson's rule to find the amount of work needed to empty all the water out over the top of the container. Your answer should be expressed in terms of $x_0, x_1, \ldots, x_6$. Take $g = 9.8$ m/sec^2, and take the density of water to be 1000 kg/m^3.

ANSWERS

1. (a) Substitute $u = 1/\sqrt{x}$, so that $du = -dx/(2x\sqrt{x})$ and hence

$$\int \frac{\cos(1/\sqrt{x})}{x^2} dx = -2 \int u \cos(u) du.$$

Using integration by parts, we get $-2u\sin(u) + 2\int \sin(u)du = -2u\sin(u) - 2\cos(u) + C = -\frac{2\sin(1/\sqrt{x})}{\sqrt{x}} - 2\cos(1/\sqrt{x}) + C$.

(b) Complete the square to get $1-(x-1)^2$ under the square root. Set up a triangle with 1 on the hypotenuse and $x-1$ on the opposite side. Then $x = 1+\sin(\theta)$, $dx = \cos(\theta)d\theta$, and $\sqrt{2x-x^2} = \cos(\theta)$. Hence

$$\int \frac{x\,dx}{(2x-x^2)^{3/2}} = \int \frac{(1+\sin(\theta))\cos(\theta)d\theta}{\cos^3(\theta)} = \int \left(\frac{1}{\cos^2(\theta)} + \frac{\sin(\theta)}{\cos^2(\theta)}\right) d\theta.$$

The first term is $\sec^2(\theta)$ with anti-derivative $\tan(\theta)$, and the second term is $\tan(\theta)\sec(\theta)$ with anti-derivative $\sec(\theta)$ (or alternatively we can do a u-substitution $u = \cos(\theta)$ to get the same thing). Putting this together, we get $\tan(\theta)+\sec(\theta)+C = \frac{x-1}{\sqrt{2x-x^2}} + \frac{1}{\sqrt{2x-x^2}} + C = \frac{x}{\sqrt{2x-x^2}} + C$.

2. The irreducible quadratic factor of x^3-1 is x^2+x+1. We need to find a, b, c such that

$$\frac{7x^2+8}{(x-1)(x^2+x+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}.$$

Clearing denominators, we get $7x^2+8 = a(x^2+x+1)+(bx+c)(x-1)$. Equating constant terms, we get $a-c=8$; equating x-terms, we get $a+c-b=0$; and equating x^2-terms, we get $a+b=7$. Replacing b by $a+c$ from the second equation, we rewrite the third equation as $2a+c=7$, which we add to the first equation. The result is $3a=15$, or $a=5$. Then $c = 7-2a = -3$ and $b = a+c = 2$, so we get $\frac{5}{x-1} + \frac{2x-3}{x^2+x+1}$.

3. Substituting $u = e^{x/2}$, so that $du = \frac{1}{2}e^{x/2}dx$, we get

$$\int_0^\infty \frac{e^{x/2}dx}{1+e^x} = 2\int_1^\infty \frac{du}{1+u^2} = 2 \lim_{b\longrightarrow\infty} \operatorname{Arctan}(u)\Big|_1^b = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}.$$

.

4. $9800\pi \int_2^5 (5-y)(1+f(y))^2 dy \approx 9800\pi \cdot \frac{1}{6}(3(1+x_0)^2 + 4\cdot 2.5(1+x_1)^2 +$
$= 2\cdot 2(1+x_2)^2 + 4\cdot 1.5(1+x_3)^2 + 2\cdot 1(1+x_4)^2 + 4\cdot 0.5(1+x_5)^2 + 0\cdot(1+x_6)^2) =$
$= \frac{9800\pi}{6}(3(1+x_0)^2 + 10(1+x_1)^2 + 4(1+x_2)^2 + 6\cdot(1+x_3)^2 + 2(1+x_4)^2 + 2(1+x_5)^2).$