1. Evaluate the following integrals. Show work. Simplify and BOX your final answer.

(a) (6 points) $\int_0^{1/2} \frac{e^{2t}}{e^{4t}+3}\, dt$

$\boxed{\begin{array}{l} u = e^{2t} \\ du = 2e^{2t}dt \end{array}}$ [Bounds: $t=0 \Rightarrow u = e^0 = 1$; $t = 1/2 \Rightarrow u = e$]

$$= \frac{1}{2}\int_1^e \frac{1}{u^2+3}\, du$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_1^e$$

$$= \frac{1}{2\sqrt{3}}\left[\arctan\left(\frac{e}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \boxed{\frac{1}{2\sqrt{3}}\left[\arctan\left(\frac{e}{\sqrt{3}}\right) - \frac{\pi}{6}\right]}$$

(b) (6 points) $\int \frac{25}{x^3+5x}dx = \int \frac{25}{x(x^2+5)}dx = \int \frac{5}{x}dx + \underbrace{\int \frac{-5x}{x^2+5}dx}$

Partial Fractions:

$$\frac{25}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$$

$$25 = (A+B)x^2 + Cx + 5A$$

$$\therefore \begin{cases} 5A = 25 \Rightarrow \boxed{A=5} \\ \boxed{C=0} \\ A+B=0 \Rightarrow \boxed{B=-5} \end{cases}$$

$\boxed{\begin{array}{l} u = x^2+5 \\ \frac{1}{2}du = x\,dx \end{array}}$

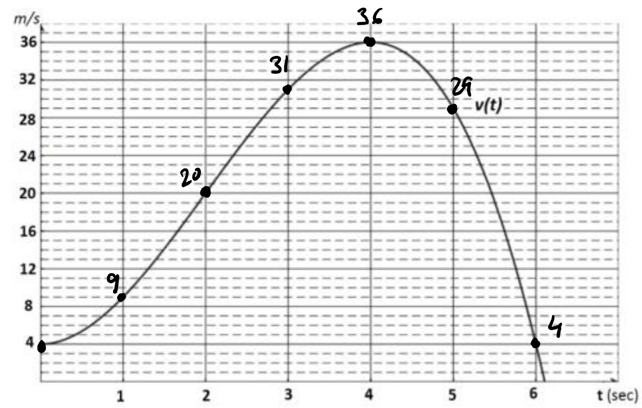
$$= 5\ln|x| + \frac{1}{2}\int \frac{-5}{u}du$$

$$= 5\ln|x| - \frac{5}{2}\ln|u| + C$$

$$= \boxed{5\ln|x| - \frac{5}{2}\ln(x^2+5) + C}$$

2. (6 points) The graph on the right shows the velocity $v(t)$ in m/s at t seconds, of an object moving in a straight line.

   Use Simpson's Rule with $n = 6$ subintervals to approximate the distance the object travels from $t = 0$ to $t = 6$ seconds.



Distance $= \int_0^6 v(t)\,dt$

$\Delta t = \frac{6-0}{6} = 1$

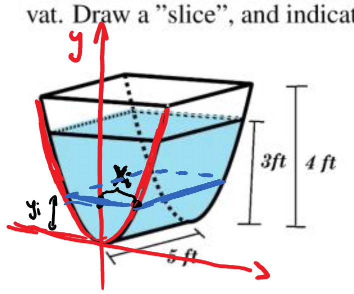
$$S_6 = \frac{1}{3}\left[v(0) + 4v(1) + 2v(2) + 4v(3) + 2v(4) + 4v(5) + v(6)\right]$$

$$= \frac{1}{3}\left[4 + 36 + 40 + 124 + 72 + 116 + 4\right]$$

$$= \frac{1}{3}[396] = \boxed{132 \text{ m}}$$

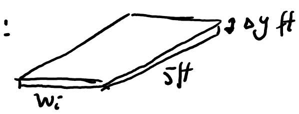
3. (7 points) A vat is of the shape shown below. The vertical ends of the vat are bounded below by the curve $y = x^2$. The vat is of length 5 feet, height 4 feet, and it is partially filled with olive oil, to a level 3 feet above its bottom. Oil weighs 50 lbs/ft^3.

   SET UP (do NOT compute) an integral equal to the work required to pump all the oil to the top of the vat. Draw a "slice", and indicate the main steps in your process of setting up the integral.



Divide $[0,3]$ into n subintervals, each of length Δy

The i^{th} "slice" of water:



volume $V_i = 5 w_i \,\Delta y$ & $w_i = 2x_i = 2\sqrt{y_i}$

$\therefore V_i = 10\sqrt{y_i}\,\Delta y$ ft^3

Weight: $F_i = 50\, V_i$ lbs $= 500\sqrt{y_i}\,\Delta y$ lbs.

Distance to be lifted: $d_i = 4 - y_i$ ft

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F_i d_i = \boxed{\int_0^3 500\sqrt{y}\,(4-y)\,dy}$$

4. (a) (8 points) Compute the average value of the following function over the interval $[0, \pi/4]$:

$$f(x) = \frac{x\sin(x)}{\cos^3(x)}$$

$$f_{ave} = \frac{1}{(\pi/4)} \int_0^{\pi/4} \frac{x\sin(x)}{\cos^3(x)}\,dx$$

The integral $\int \frac{x \sin x}{\cos^3 x}\,dx$ can be computed via integration by parts:

$$\boxed{\begin{array}{ll} u = x & du = dx \\ dv = \frac{\sin x}{\cos^3 x} & v = \int \frac{\sin x}{\cos^3 x}\,dx = \frac{1}{2}\sec^2 x + C \end{array}}$$

various substitutions work:

$w = \cos x \quad dw = -\sin x\,dx$
OR
$w = \sec x \quad dw = \tan x \sec x\,dx$
OR
$w = \tan x \quad dw = \sec^2 x\,dx$

The last one gives:
$v = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C$
$= \frac{1}{2}\sec^2 x + C$

$$\therefore \int \frac{x\sin x}{\cos^3 x}\,dx = \frac{1}{2}x\sec^2 x - \frac{1}{2}\int \sec^2 x\,dx$$

$$= \boxed{\frac{1}{2}x\sec^2 x - \frac{1}{2}\tan x + C}$$

$$\therefore f_{ave} = \frac{\cancel{4}\,2}{\pi}\left[\frac{1}{\cancel{2}}x\sec^2 x - \frac{1}{\cancel{2}}\tan x\right]\Bigg|_0^{\pi/4} = \frac{2}{\pi}\left[\frac{\pi}{4}\cdot 2 - 1\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2} - 1\right) = \boxed{1 - \frac{2}{\pi}} = \boxed{\frac{\pi - 2}{\pi}}$$

(b) (2 points) Suppose $g(x)$ is some continuous function and its average value on $[a, b]$ is g_{ave}. Circle the correct answer. You need not justify.

$\underbrace{\int_a^b (g(x) - g_{ave})\,dx}$ is: A) > 0, B) $= 0$, C) < 0, D) It depends.

$= \int_a^b g(x)\,dx - g_{ave}(b-a)$ OR think of the geometric interpretation of g_{ave}

5. (7 points) Does the following improper integral converge or diverge? Note that this integral is both Type I and Type II. If it converges, compute its value. If it diverges, show why. Make sure to use limits and show all steps.

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{-u}\,du = -2e^{-u} = -2e^{-\sqrt{x}} = -\frac{2}{e^{\sqrt{x}}} + C$$

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx + \int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+}\int_a^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx + \lim_{b\to\infty}\int_1^b \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+}\left[\frac{-2}{e^{\sqrt{x}}}\right]\Big|_a^1 + \lim_{b\to\infty}\left[\frac{-2}{e^{\sqrt{x}}}\right]\Big|_1^b$$

$$= \left(-\frac{2}{e} + \frac{2}{e^0}\right) + \underbrace{\lim_{b\to\infty}\left(-\frac{2}{e^{\sqrt{b}}}\right.}_{\text{"}\frac{2}{\infty}\text{"} = 0.} \left. + \frac{2}{e^1}\right)$$

$$= -\cancel{\frac{2}{e}} + 2 + 0 + \cancel{\frac{2}{e}}$$

$$= \boxed{2}$$

6. (8 points) Evaluate the following integral. Simplify and box your answer.

$$\int \sqrt{5-4x-x^2}\,dx$$

Complete the square: $-x^2-4x+5 = -(x^2+4x-5)$

$$= -\left[(x+2)^2-4-5\right]$$

$$= 9-(x+2)^2$$

$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-(x+2)^2}\,dx$$

TRIG SUB: $x+2 = 3\sin\theta$

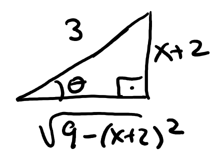
$dx = 3\cos\theta\,d\theta$

$$= \int \sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta$$

$$= \int 9\cos^2\theta\,d\theta$$

$$= 9\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$



$$= \frac{9}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{9}{2}\left[\arcsin\left(\frac{x+2}{3}\right) + \frac{x+2}{3}\,\frac{\sqrt{9-(x+2)^2}}{3}\right] + C$$

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x+2}{3}\right) + \frac{1}{2}(x+2)\sqrt{5-4x-x^2} + C}$$