Mid 2 Solutions

1. (14 points) Evaluate the following integrals. Show all your work and box your final answer.

(a) $\int \cos^4(x)\,dx$

$$= \int \left(\cos^2(x)\right)^2 dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int (1+\cos(2x))^2 dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \underbrace{\cos^2(2x)}\, dx =$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\, dx$$

$$= \frac{1}{4}\left[x + \cancel{2}\frac{\sin(2x)}{\cancel{2}} + \frac{1}{2}\left(x + \frac{\sin(4x)}{4}\right)\right] + C = \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin(4x)\right]$$

$$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$$

(b) $\int \frac{e^x}{e^{2x} + 8e^x + 12}\,dx$ $\longrightarrow$ 1) u-sub: $u = e^x$, $du = e^x dx$

$$= \int \frac{1}{u^2 + 8u + 12}\,du$$ $\longrightarrow$ 2) Partial Fractions:

$$\frac{1}{u^2+8u+12} = \frac{1}{(u+2)(u+6)} = \frac{A}{u+2} + \frac{B}{u+6}$$

$$1 = A(u+6) + B(u+2)$$

$u=-2$: $1 = 4A \Rightarrow A = 1/4$

$u=-6$: $1 = B(-4) \Rightarrow B = -1/4$.

$$= \int \frac{1/4}{u+2} - \frac{1/4}{u+6}\,du$$

$$= \frac{1}{4}\int \frac{1}{u+2} - \frac{1}{u+6}\,du$$

$$= \frac{1}{4}\left(\ln|u+2| - \ln|u+6|\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{u+2}{u+6}\right| + C$$

$$= \boxed{\frac{1}{4}\ln\left[\frac{e^x+2}{e^x+6}\right] + C}$$

2. (8 points) For each of the four integrals below, just state which of the following methods applies. Your answer should be in one of the following forms:

- u-substitution, with $u = ...$ (*specify the substitution to use*)
- integration by parts, with $u = ...$, and $dv = ...$ (*specify the parts to use*)
- trigonometric substitution, with $x = ...$ (*specify the trig sub to apply*)
- partial fractions, with fractions: $\frac{A}{(...)} + ...$ (*specify the general decomposition*)

You do not need to justify or compute anything – and do not evaluate the integrals!

(a) $\int x^2 \ln(x)\, dx$ Method: Integration by Parts

With: $u = \ln x$ $dv = x^2\, dx$.

(b) $\int x^2 \sec^2(x^3)\, dx$ Method: u-substitution

with: $u = x^3$

(c) $\int \frac{-2x+1}{x^4+x^3+x^2}\, dx$ Method: Partial Fractions

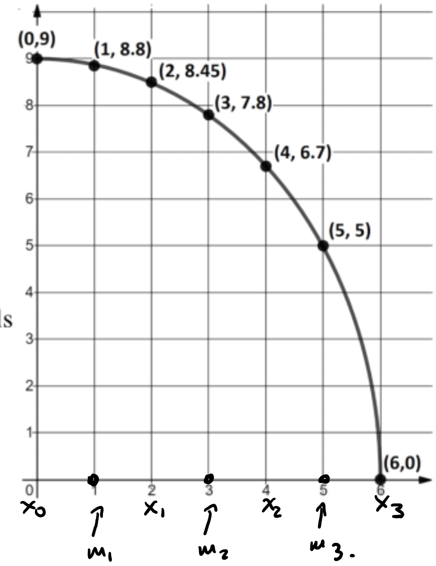
With: $\frac{A}{x} + \frac{B}{x^2} + \frac{(x+1)}{x^2+x+1}$

$\int \frac{-2x+1}{x^2(x^2+x+1)}$

(d) $\int \frac{x^2}{(x^2-4)^{3/2}}\, dx$ Method: Trigonometric Substitution

with: $x = 2\sec\theta$

3. The graph of a function $y = f(x)$ is shown on the right, together with the (x, y)-coordinates of selected points on its graph.



Estimate the **average value** f_{ave} of this function over the interval $[0, 6]$, as follows:

(a) (4 points) By the Midpoint Rule with $n = 3$ subintervals

$\boxed{f_{ave} = \frac{1}{6}\int_0^6 f(x)\,dx}$ $\Delta x = \frac{6}{3} = 2$

$m_1 = 1,\ m_2 = 3,\ m_3 = 5$.

$M_3 = \frac{1}{6}\Delta x\left[f(m_1) + f(m_2) + f(m_3)\right]$

$= \frac{1}{6}\cdot 2\ [8.8 + 7.8 + 5]$

$= \frac{1}{3}[21.6] = \boxed{7.2}$

(b) (4 points) Using the Trapezoidal Rule with $n = 3$ subintervals

$T_3 = \frac{1}{6}\left[\frac{\Delta x}{2}\left(f(0) + 2f(2) + 2f(4) + f(6)\right)\right]$

$= \frac{1}{6}\left[\frac{2}{2}\left(9 + 2(8.45) + 2(6.7) + 0\right)\right]$

$= \frac{1}{6}(9 + 16.9 + 13.4) = \frac{1}{6}(39.3) = \boxed{6.55}$.

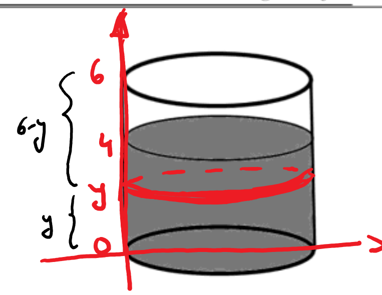
(c) (2 points) How does the average value f_{ave} on [0, 6] compare to its Trapezoidal and Midpoint approximations? *Circle one of the answers AND one of the justifications below.*

- f_{ave} is higher than both T_n and M_n
- f_{ave} is lower than both T_n and M_n
- $T_n < f_{ave} < M_n$
- cannot tell

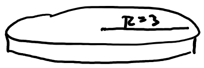
Circle a justification for your answer:

- Because the function f is non-negative on $[0, 6]$
- Because the function f is concave-down on $[0, 6]$
- Because the function f decreases on $[0, 6]$
- It depends on n

4. (8 points) A cylindrical tank has radius 3 ft and it's 6 feet tall. The tank is partly full with oil, to a height of 4 feet, as shown. The oil in the tank weighs 50 lbs/ft^3.



(a) Consider a **thin horizontal layer** of oil, of thickness Δy, that is **at** y **ft from the bottom of the tank**. Write an expression in y and Δy that is approximately equal to the **work**, in ft-lbs, required to lift just this thin horizontal layer to the top of the tank.



$\Delta y \Rightarrow$ Volume layer $= \pi (3)^2 \Delta y = 9\pi \Delta y$ ft^3

$\therefore$ Force for layer $= 50\,(9\pi \Delta y) = 450\pi \Delta y$ lbs.

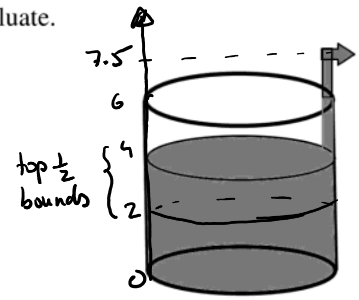
distance to lift $\approx 6-y$

$\therefore$ Work for layer $\approx \boxed{450\pi(6-y)\,\Delta y}$ ft-lbs.

(b) Set up an integral in y equal to the work required to pump all the oil in the tank to the top of the tank. Do not evaluate the integral.

$$W_{total} = \int_0^4 450\pi(6-y)\,dy$$

(c) Set up an integral equal to the work required to pump only the top **half** of the oil in the tank to a spout that's 1.5 feet above the top of the tank. Do not evaluate.



$$W_{top\frac{1}{2}} = \int_2^4 450\pi\,(7.5-y)\,dy$$

5. (10 points) Evaluate the following improper integral. Make sure to use limits and show all your work.

$$\int_0^{\infty} \frac{1}{(\sqrt{x^2+4x+5})^3}\,dx$$

Type I only

① Indefinite Integral (antiderivative)

$$\int \frac{1}{(\sqrt{x^2+4x+5})^3}\,dx = \int \frac{1}{(\sqrt{(x+2)^2+1})^3}\,dx$$

TRIG SUB:

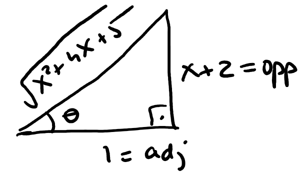
$$\boxed{\begin{aligned} x+2 &= \tan\theta \\ dx &= \sec^2\theta\, d\theta \end{aligned}}$$

$$= \int \frac{1}{(\sqrt{\tan^2\theta+1})^3}\sec^2\theta\,d\theta$$

$$= \int \frac{1}{(\sec\theta)^3}\sec^2\theta\,d\theta = \int \frac{1}{\sec\theta}\,d\theta$$

$$= \int \cos\theta\,d\theta = \sin\theta + C$$

$$= \frac{x+2}{\sqrt{x^2+4x+5}} + C$$



$$\sin\theta = \frac{x+2}{\sqrt{\cdots}}$$

② $$\int_0^{\infty} \frac{1}{(\sqrt{x^2+4x+5})^3}\,dx = \lim_{t\to\infty}\left(\frac{x+2}{\sqrt{x^2+4x+5}}\Bigg|_0^t\right)$$

$$= \underbrace{\lim_{t\to\infty}\left(\frac{t+2}{\sqrt{t^2+4t+5}}\right.}_{\downarrow} \left. - \frac{2}{\sqrt{5}}\right)$$

$$= \boxed{1 - \frac{2}{\sqrt{5}}}$$

$$\lim_{t\to\infty} \frac{t+2}{\sqrt{t^2+4t+5}}$$

$$= \lim_{t\to\infty} \frac{1+2/t}{\sqrt{1+4/t+5/t^2}}$$

$$= \frac{1+0}{\sqrt{1+0+0}} = 1$$

Integral converges to $1-\frac{2}{\sqrt{5}}$.