MIDTERM #2

Math 125H

name TA section

You must show all work for full credit. Use the backs of the test pages as necessary. Give all answers in EXACT FORM (no decimal approximations).

1. Evaluate the following integrals.

a. $\displaystyle\int \frac{2x+3}{x^2-2x+1}\,dx = \int \frac{2(x-1)+5}{(x-1)^2}\,dx = 2\ln|x-1| - \frac{5}{x-1} + C$

b.
$$\begin{aligned}\int_0^{\pi/4} \cos^6(3x)\sec^3(3x)\,dx &= \int_0^{\pi/4} \cos^3(3x)\,dx \\ &= \int_0^{\pi/4} \cos(3x)\left(1-\sin^2(3x)\right)\,dx \\ &= \int_0^{\sqrt{2}/2} \frac{1}{3}\left(1-u^2\right)\,du \\ &= \left(\frac{1}{3}u - \frac{1}{9}u^3\right)\Bigg|_0^{\sqrt{2}/2} = \frac{5\sqrt{2}}{36}\end{aligned}$$

2. A tank of liquid has the shape of the top half of a sphere of radius 4 feet. It has a spigot located at its topmost point. Find the work done in pumping the liquid out of the tank. Assume that the liquid has density 1 pound per cubic foot.

$$\begin{aligned}\text{Work} &= \int_0^4 \pi(4-h)\left(16-h^2\right)\,dh = \int_0^4 \pi\left(h^3-4h^2-16h+64\right)\,dh \\ &= \pi\left(\frac{h^4}{4} - \frac{4h^3}{3} - 8h^2 + 64h\right)\Bigg|_0^4 \\ &= \pi\left(\frac{704}{3} - 128\right) = \frac{320\pi}{3}\ \text{ft} - \text{lbs}\end{aligned}$$

3. A certain function $f(x)$ is defined on the interval $[1,3]$. Show that the length of its graph is at least 2.

$$\text{Arclength} = \int_1^3 \sqrt{1+f'(x)^2}\,dx \geq \int_1^3 1\,dx = 2.$$

4. Show that $\int_0^\infty e^{-x^3}\sin^2 x\,dx$ converges. (Do not try to evaluate it.)

$$\int_0^\infty e^{-x^3}\sin^2 x\,dx = \int_0^1 e^{-x^3}\sin^2 x\,dx + \int_1^\infty e^{-x^3}\sin^2 x\,dx$$

The first integral is proper. The second is bounded by the convergent integral $\int_1^\infty e^{-x}\,dx$. Thus the integral converges.

5. Evaluate $\int_{1/2}^1 \frac{\sqrt{1-x^2}}{x}\,dx$. You may find it helpful to recall that $\int \sec x = \ln|\sec x + \tan x| + c$, or that $\int \csc x = -\ln|\csc x + \cot x| + c$.

First use the substitution $x = \cos u$ and $dx = -\sin u\,du$.

$$\begin{aligned}
\int_{1/2}^1 \frac{\sqrt{1-x^2}}{x}\,dx &= \int_{\pi/3}^0 -\frac{\sin^2 u}{\cos u}\,du = \int_0^{\pi/3} \frac{\sin^2 u}{\cos u}\,du \\
&= \int_0^{\pi/3} (\sec u - \cos u)\,du = \left(\ln|\sec u + \tan u| - \sin u\right)\Big|_0^{\pi/3} \\
&= \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}
\end{aligned}$$