University of Washington **Department of Mathematics**

Midterm Exam #2 - Solutions **Math 125E** **February 27, 2003**

1. (a) (5 points) Use the substitution $u = \sin x$, so that $du = \cos x\, dx$. This gives

$$\int \frac{\cos x}{4 - \sin^2 x}\, dx = \int \frac{1}{4 - u^2}\, du = -\int \frac{1}{(u-2)(u+2)}\, du.$$

Using partial fractions this becomes

$$\begin{aligned}\int \frac{1}{4-u^2}\, du &= -\int \left(\frac{1}{4}\frac{1}{u-2} - \frac{1}{4}\frac{1}{u+2}\right) du \\ &= -\left(\frac{1}{4}\ln|u-2| - \frac{1}{4}\ln|u+2|\right) + C \\ &= \frac{1}{4}\ln|\sin x + 2| - \frac{1}{4}\ln|\sin x - 2| + C.\end{aligned}$$

(b) (5 points) Let $x = 2\sin\theta$; then $dx = 2\cos\theta\, d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{4-x^2}}\, dx &= 8\int \sin^3\theta\, d\theta \\ &= 8\int (1-\cos^2\theta)\sin\theta\, d\theta \\ &= 8[-\cos\theta + \frac{1}{3}\cos^3\theta] + C \\ &= -4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + C.\end{aligned}$$

2. (a) (5 points) Integrate by parts, letting $u = x^2 + 1$ and $dv = e^{-x}\, dx$ so that $du = 2x\, dx$ and $v = -e^{-x}$. This gives

$$\int_0^1 (x^2+1)e^{-x}\, dx = [-(x^2+1)e^{-x}]_0^1 + \int_0^1 2xe^{-x}\, dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\, dx.$$

Integrate by parts again, now letting $u = x$ and again $dv = e^{-x}\, dx$ so that now $du = dx$ and $v = -e^{-x}$. This gives

$$\int_0^1 xe^{-x}\, dx = [-xe^{-x}]_0^1 + \int_0^1 e^{-x}\, dx = -2e^{-1} + 1.$$

Combining the two yields

$$\int_0^1 (x^2+1)e^{-x}\, dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -6e^{-1} + 3.$$

(b) (5 points) Let $u = 1/x$, then $du = -dx/x^2$ so

$$\int_1^4 \frac{e^{1/x}}{x^2}\, dx = \int_{1/4}^1 e^u \, du = [e^u]_{1/4}^1 = e - e^{1/4}.$$

3. (10 points) The region under the curve $y = \cos^2 x$ for $0 \le x \le \pi/2$ is rotated about the x-axis. Find the volume of the resulting solid.

Slicing parallel to the x-axis we see that the slices are disks of radius $[\cos^2 x]^2$, so the volume is

$$\begin{aligned} V &= \int_0^{\pi/2} \pi[f(x)]^2\, dx = \int_0^{\pi/2} \pi[\cos^2 x]^2\, dx = \pi \int_0^{\pi/2} [\frac{1}{2}(1+\cos 2x)]^2\, dx \\ &= \frac{\pi}{4}\int_0^{\pi/2} [1+\cos^2 2x + 2\cos 2x]\, dx = \frac{\pi}{4}\int_0^{\pi/2} [1+\frac{1}{2}(1+\cos 4x) + 2\cos 2x]\, dx \\ &= \frac{\pi}{4}\left[\frac{3}{2}x + \frac{1}{2}\left(\frac{1}{4}\sin 4x\right) + 2\left(\frac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \frac{\pi}{4}\left[\left(\frac{3\pi}{4} + \frac{1}{8}\cdot 0 + 0\right) - 0\right] = \frac{3\pi^2}{16}. \end{aligned}$$

4. (10 points) Suppose that at time $t = 10$ seconds an object is traveling at 30.0 m/sec. Its acceleration $a(t)$ is measured at two-second intervals until time $t = 20$, with the following results (the units of acceleration are m/sec^2):

t	10	12	14	16	18	20
$a(t)$	2.3	2.4	2.5	2.6	2.6	2.7

Use the trapezoidal rule with $n = 5$ to estimate the velocity of the object at time $t = 20$.

Recall that acceleration is the derivative of velocity, so

$$v(20) = v(10) + \int_{10}^{20} a(t)\, dt$$

Using the trapezoidal rule with $n = 5$ to estimate this definite integral we have $\Delta x = 2$ so that

$$\begin{aligned} \int_{10}^{20} a(t)\, dt &\approx \frac{2}{2}[a(10) + 2a(12) + 2a(14) + 2a(16) + 2a(18) + a(20)] \\ &= [2.3 + 2(2.4) + 2(2.5) + 2(2.6) + 2(2.6) + 2.7] = 25.2 \end{aligned}$$

So since $v(10) = 30.0$ we conclude that

$$V(20) \approx 30.0 + 25.2 = 55.2 m/sec.$$

5. (10 points) The gamma function is defined for all $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

(a) (6 points) Find $\Gamma(1)$ and $\Gamma(2)$.

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = \lim_{s\to\infty} \int_0^s e^{-t}\, dt = \lim_{s\to\infty} [-e^{-t}]_0^s = \lim_{s\to\infty} [-e^{-s} + 1] = 1.$$

and

$$\Gamma(2) = \int_0^\infty t^{2-1} e^{-t}\, dt = \int_0^\infty t e^{-t}\, dt = \lim_{s\to\infty} \int_0^s t e^{-t}\, dt.$$

Integrating by parts, with $u = t$ and $dv = e^{-t}\, dt$ so that $du = dt$ and $v = -e^{-t}$, we see that

$$\int_0^s t e^{-t}\, dt = [-te^{-t}]_0^s + \int_0^s e^{-t}\, dt = -se^{-s} - e^{-s} + 1.$$

Since $\lim_{s\to\infty} se^{-s} = 0$ we have

$$\Gamma(2) = \lim_{s\to\infty} (-se^{-s} - e^{-s} + 1) = 1 \qquad \text{also.}$$

(b) (4 points) Use integration by parts to show that, for positive n

$$\Gamma(n+1) = n\Gamma(n).$$

When $x = n+1$,

$$\Gamma(n+1) = \int_0^\infty t^{(n+1)-1} e^{-t}\, dt = \int_0^\infty t^n e^{-t}\, dt = \lim_{s\to\infty} \int_0^s t^n e^{-t}\, dt.$$

By using integration by parts as above, now with $u = t^n$ one can easily see that

$$\int t^n e^{-t}\, dt = -t^n e^{-t} + n \int t^{(n-1)} e^{-t}\, dt$$

(see also problem 7.1 # 42 which is similar). So

$$\Gamma(n+1) = \lim_{s\to\infty} \left([-t^n e^{-t}]_0^s + \int_0^s t^{(n-1)} e^{-t}\, dt \right) = \lim_{s\to\infty} \left([0 + s^n e^{-s}] + \int_0^s t^{(n-1)} e^{-t}\, dt \right).$$

The key point now is that $\lim_{s\to\infty} s^n e^{-s} = 0$ since the exponential function decays faster than any polynomial, so

$$\Gamma(n+1) = \lim_{s\to\infty} \int_0^s t^{(n-1)} e^{-t}\, dt = \int_0^\infty t^{(n-1)} e^{-t}\, dt = \Gamma(n).$$