1. Compute the following definite integrals. Simplify, but leave your answers in exact form.

(a) [**6 points**] $\int_{\pi/4}^{\pi/3} \tan^3(\theta)\sec^2(\theta)\,d\theta$

Sol #1: $u = \tan\theta$, $du = \sec^2\theta\, d\theta$

$$\int_{\pi/4}^{\pi/3} \tan^3(\theta)\sec^2(\theta)\,d\theta$$

$$= \int_1^{\sqrt{3}} u^3\,du = \frac{u^4}{4}\Big|_1^{\sqrt{3}}$$

$$= \frac{9}{4} - \frac{1}{4} = \boxed{2}$$

Sol #2: $u = \sec\theta$ $du = \sec\theta\tan\theta\,d\theta$

$$\int_{\pi/4}^{\pi/3} \underbrace{\tan^2(\theta)}_{\sec^2\theta - 1}\sec(\theta)\ \sec(\theta)\tan(\theta)\,d\theta$$

$$= \int_{\sqrt{2}}^{2} (u^2-1)\,u\,du = \int_{\sqrt{2}}^{2} u^3 - u\,du$$

$$= \left(\frac{u^4}{4} - \frac{u^2}{2}\right)\Big|_{\sqrt{2}}^{2} = \left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{4}{4} - \frac{2}{2}\right)$$

$$= \boxed{2}$$

(b) [**6 points**] $\int_1^2 x^3 \ln x\,dx$

Integration by Parts: $u = \ln x$ $dv = x^3$
$du = \frac{1}{x}dx$ $v = \frac{1}{4}x^4$

$$\int_1^2 x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x\Big|_1^2 - \int_1^2 \frac{1}{4}x^4 \frac{1}{x}\,dx$$

$$= \left(\frac{1}{4}16\ln 2 - \frac{1}{4}\underbrace{\ln 1}_{0}\right) - \frac{1}{4}\int_1^2 x^3\,dx$$

$$= 4\ln 2 - \frac{1}{4}\frac{x^4}{4}\Big|_1^2 = 4\ln 2 - \frac{1}{16}(16-1)$$

$$= \boxed{4\ln 2 - \frac{15}{16}}$$

Other methods might work too. For ex: $u = \ln x$, $du = \frac{1}{x}dx \Rightarrow dx = x\,du$ & $x = e^u$
results in $\int_0^{\ln 2} (e^u)^4 u\,du = \int_0^{\ln 2} u e^{4u}\,du$
and now one can apply IBP w/ $w = u$ $dv = e^{4u}$
$dw = du$ $v = \frac{1}{4}e^{4u}$ ...

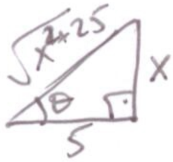
2. Evaluate the following indefinite integrals.

(a) [**6 points**] $\int \frac{1}{x^2\sqrt{x^2+25}}dx = \int \frac{1}{25\tan^2\theta\sqrt{25\tan^2\theta+25}} 5\sec^2\theta\, d\theta.$

Trig Sub: $\boxed{\begin{array}{l} x = 5\tan\theta \\ dx = 5\sec^2\theta\, d\theta \end{array}}$ $= \int \frac{1}{25\tan^2\theta(\cancel{5}\cancel{\sec}\theta)} \cancel{5}\sec^{\cancel{2}}\theta\, d\theta$

$= \frac{1}{25}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}d\theta$ $\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}}$

$= \frac{1}{25}\int \frac{1}{u^2}du = \frac{1}{25}\left(-\frac{1}{u}\right) + C = -\frac{1}{25\sin\theta} + C$



$\sin\theta = \frac{x}{\sqrt{x^2+25}}$

so: $\frac{1}{\sin\theta} = \frac{\sqrt{x^2+25}}{x}.$

$= \boxed{-\frac{\sqrt{x^2+25}}{25x} + C}.$

(b) [**6 points**] $\int \frac{2x^3+1}{x^3-x^2}\,dx = \int 2 + \frac{2x^2+1}{x^3-x^2}dx = 2x + \int \frac{2x^2+1}{x^2(x-1)}dx$

P.F.: $\frac{2x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$2x^2+1 = Ax(x-1) + B(x-1) + Cx^2$

$x=0$: $1 = B(-1) \Rightarrow B = -1$

$x=1$: $3 = C(1) \Rightarrow C = 3$

$x=-1$: $\cancel{3} = 2A - 1(-2) + \cancel{3} \Rightarrow A = -1$

$= 2x + \int \frac{-1}{x} - \frac{1}{x^2} + \frac{3}{x-1}dx$

$= \boxed{2x - \ln|x| + \frac{1}{x} + 3\ln|x-1| + C}$

$= 2x + \frac{1}{x} + \ln\left|\frac{(x+1)^3}{x}\right| + C$

$= \frac{2x^2+1}{x} + \ln\left|\frac{(x+1)^3}{x}\right| + C$

Division: $\frac{2x^3+1}{x^3-x^2} = \frac{2(x^3-x^2)+2x^2+1}{x^3-x^2} = 2 + \frac{2x^2+1}{x^3-x^2}$

(OR: $\begin{array}{r|l} & 2 \\ x^3-x^2 & 2x^3+1 \\ & 2x^3-2x^2 \\ \hline & 2x^2+1 \end{array}$)

3.

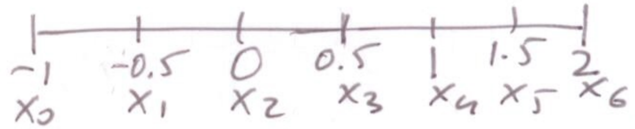
(a) [**3 points**] Write an integral expression to compute the average value of $f(x) = \cos(x^2)$ over the interval $[-1, 2]$. DO NOT try to compute the integral.

$$f_{ave} = \frac{1}{3}\int_{-1}^{2} \cos(x^2)\,dx$$

(b) [**5 points**] Use the Trapezoidal Rule with $n = 6$ subintervals to approximate the integral from part (a).

Your answer should either be in exact form (but simplify all you can), or in decimal form with at least 4 digits of precision.

$$\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$$



$$f_{ave} = \frac{1}{3}\left[\frac{\frac{1}{2}}{2}\left(\cos(1) + 2\cos(0.25) + 2\cos(0) + 2\cos(0.25) + 2\cos(1) + 2\cos(2.25) + \cos(4)\right)\right]$$

$$= \frac{1}{3}\left[\frac{1}{4}\left(3\cos(1) + 4\cos(0.25) + 2 + 2\cos(2.25) + \cos(4)\right)\right]$$

$$\cong \frac{1}{3}\left[\frac{1}{4}(5.5865657\ldots)\right]$$

$$= \frac{1}{3}\left[1.3966414\ldots\right]$$

$$= \boxed{0.4655}47\ldots$$

4. **[9 points]** Consider the improper integral: $\int_0^\infty \frac{1}{e^x+1}\,dx$.

If it converges, evaluate it. If it diverges, show why. Show all your steps carefully.

$$\int_0^\infty \frac{1}{e^x+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{e^x+1}\,dx$$

u-sub.

$$\boxed{\begin{aligned} u &= e^x+1 \\ du &= e^x\,dx \\ &= (u-1)\,dx \end{aligned}}$$

bounds:
$x=0 \Rightarrow u=2$
$x=b \Rightarrow u=e^b+1$
Take $c=e^b+1$

$$= \lim_{c\to\infty}\int_2^c \frac{1}{u}\,\frac{1}{u-1}\,du$$

$$= \lim_{c\to\infty}\int_2^c -\frac{1}{u}+\frac{1}{u-1}\,du$$

$$= \lim_{c\to\infty}\left(-\ln|u|+\ln|u-1|\right)\Big|_2^c$$

$$= \lim_{c\to\infty}\ln\left|\frac{u-1}{u}\right|\Big|_2^c$$

$$= \lim_{c\to\infty}\ln\left|1-\frac{1}{c}\right|-\ln\left(\tfrac{1}{2}\right)$$

$$= \underbrace{\cancel{\ln} 1}_{0}-\ln\left(\tfrac{1}{2}\right) = \boxed{-\ln\tfrac{1}{2}}$$

$$= \boxed{\ln 2}.$$

$$\frac{1}{u(u-1)} = \frac{A}{u}+\frac{B}{u-1}$$

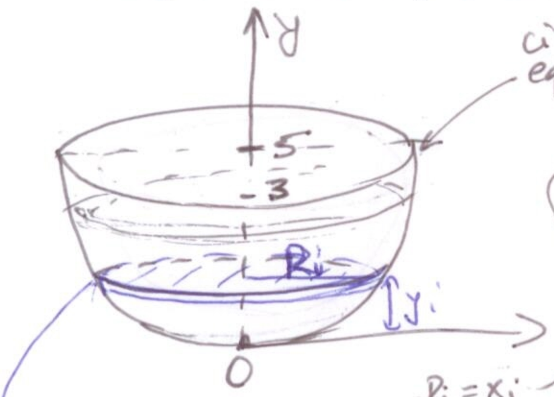
$$1 = A(u-1)+Bu$$

$$1 = (A+B)u - A \Rightarrow \begin{cases} A+B=0 \Rightarrow B=-A=1 \\ 1=-A \Rightarrow A=-1 \end{cases}$$

$$\frac{1}{u(u-1)} = \frac{-1}{u}+\frac{1}{u-1}.$$

5. [**9 points**] A tank in the shape of the bottom half of a sphere of radius $R = 5$ m is partially filled with water, to a height of 3 meters. The density of water is 1000kg/m^3 and the gravitational constant is $g = 9.8$ m/s^2. Compute the work necessary to pump all the water to the top of the tank.

(First draw a picture and clearly label your origin and axes. Make sure to show your steps clearly. Include units in your final answer.)



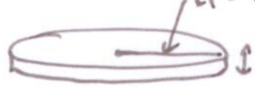
circle equation: $x^2 + (y-5)^2 = 25$

$\boxed{x^2 = 25 - (y-5)^2}$

Slice the interval $[0, 3]$ into n slices

The i^{th} slice of water, at height y_i above the bottom, is in the shape of a

$R_i = x_i$

disk →  Δy $\Rightarrow$ $V_i = \pi (R_i)^2 \Delta y = \pi (x_i^2) \Delta y$

$= \pi (25 - (y_i - 5)^2) \Delta y$ in m^3

So the force required to lift the i^{th} slice (its weight) is

$F_i = V_i (1000)(9.8) = 9800\pi (25 - (y_i-5)^2) \Delta y$ (Newtons)

The i^{th} slice must be lifted from height $= y_i$ to height $= 5$ m.

$d_i = 5 - y_i$ (meters)

Hence the work for lifting the i^{th} slice is:

$W_i = F_i d_i = 9800\pi (25 - (y_i - 5)^2)(5 - y_i) \Delta y$ Joules.

Total work: $W = \int_0^3 9800\pi (25 - (y-5)^2)(5-y)\, dy$

$= 9800\pi \int_0^3 (-y^2 + 10y)(5-y)\, dy$

multiply out & compute, or $u = -y^2 + 10y$ $du = 2(5-y)dy$

$= 9800\pi \int_0^{21} 1/2\, u\, du = 9800\pi \frac{u^2}{4} \Big|_0^{21} = 9800\pi \frac{441}{4}$

$= \boxed{1{,}080{,}450\pi \text{ Joules}}$.