Math 125

Midterm 2 (February 27, 2020)

NAME: Solutions **Section:** ____________

- Time: you have **80 minutes.**
- Please show all work and justify your answers. The final answers must be "reasonably" simplified. For example, a rational number must be given in the form $\frac{a}{b}$ for some integers a and b, but it is ok to have expressions like $\ln 3$ or e^4 in your final answer.
- You are allowed to use calculator (Model TI-30X IIS only) and one *handwritten* (with your own handwriting) 8.5 x 11 inch sheet of notes. Writing allowed on both sides.
- Have your *Husky Card* visible on the desk beside you.
- You may use both sides of the paper.
- Make sure you have **9 pages** and **6 problems** before starting the exam.

Academic integrity is expected of all students at all times. Understanding this, I declare I shall not give, use, or receive unauthorized aid.

SIGNATURE: ____________________

Problem 1:	___ / 20
Problem 2:	___ / 20
Problem 3:	___ / 20
Problem 4:	___ / 20
Problem 5:	___ / 20
Problem 6:	___ / 20

Total: ___ / 120

Problem 1: Evaluate the following integrals:

(a)

$$\int \frac{x^4}{\sqrt{(x^2+1)^7}}\,dx$$

(b)

$$\int_0^1 \tan^{-1} x\,dx$$

Recall: $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2+1}$.

a) Let $\left\{\begin{array}{l} x = \tan\theta \;, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ dx = \sec^2\theta\, d\theta \end{array}\right\}$

$\left\{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}\right\}$

$$\int \frac{x^4}{\sqrt{(x^2+1)^7}}\,dx = \int \frac{\tan^4\theta}{\underbrace{\sec^7\theta}}\sec^2\theta\, d\theta = \int \sin^4\theta\cos\theta\, d\theta = \int u^4\,du$$

$$\left(\sqrt{\sec^2\theta} = +\sec\theta \quad \text{for } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\right)$$

$$= \frac{u^5}{5} + \text{const.} = \frac{\sin^5\theta}{5} + \text{Const.} = \frac{1}{5}\left(\frac{x}{\sqrt{1+x^2}}\right)^5 + \text{Const.}$$

$$\left\{\begin{array}{l} \cos^2\theta = \dfrac{1}{1+\tan^2\theta} = \dfrac{1}{1+x^2} \\ \cos\theta = \dfrac{+1}{\sqrt{1+x^2}} \quad \left(-\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\right) \\ \sin\theta = \tan\theta\cos\theta = \dfrac{x}{\sqrt{1+x^2}} \end{array}\right\}$$

b) By parts

$$\int \underbrace{\tan^{-1}x}_{u}\,\underbrace{dx}_{dv} = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$$

$du = \frac{1}{1+x^2}dx$ $\quad$ $v = x$

$\left\{\begin{array}{l} w = 1+x^2 \\ dw = 2x\,dx \end{array}\right\}$

SO $$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x - \frac{1}{2}\ln|1+x^2|\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Problem 2: Compute the integral

$$\int \frac{x^2+1}{x^3+x^2}\,dx$$

by the *method of partial fractions.*

$$\frac{X^2+1}{X^2(X+1)} = \frac{A}{X} + \frac{B}{X^2} + \frac{C}{X+1} \rightsquigarrow X^2+1 = AX(X+1) + B(X+1) + CX^2$$

$X=0 \rightsquigarrow B=1$

$X=-1 \rightsquigarrow C=2$

$X=1 \rightsquigarrow 2 = 2A+2B+C \rightarrow A=-1$

$$\int \frac{X^2+1}{X^2(X+1)}\,dX = \int \left(\frac{-1}{X} + \frac{1}{X^2} + \frac{2}{X+1} \right) dX = -\operatorname{Ln}|X| - \frac{1}{X} + 2\operatorname{Ln}|X+1| + \text{Const.}$$

Problem 3: Determine, with justification, the *convergence* or *divergence* of each of the following improper integrals.

(a)

$$\int_{-\infty}^{+\infty} xe^{-x^2}\, dx$$

(b)

$$\int_{1}^{+\infty} \frac{x}{\sqrt{x+x^6}}\, dx$$

a) $\int_{-\infty}^{+\infty} xe^{-x^2} dx$ converges because both $\int_{-\infty}^{0} xe^{-x^2} dx$ and $\int_{0}^{+\infty} xe^{-x^2} dx$ converge.

$\int_{0}^{+\infty} xe^{-x^2} dx = \frac{1}{2}\left[-e^{-x^2}\right]_{0}^{+\infty} = \frac{1}{2}$, $\int_{-\infty}^{0} xe^{-x^2} dx = \frac{1}{2}\left[-e^{-x^2}\right]_{-\infty}^{0} = -\frac{1}{2}$

In fact: $\int_{-\infty}^{+\infty} xe^{-x^2} dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$

b) $\sqrt{x+x^6} > \sqrt{x^6}$ for $x \geq 1$, So $\frac{x}{\sqrt{x+x^6}} < \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$

So $\int_{1}^{+\infty} \frac{x}{\sqrt{x+x^6}} dx < \int_{1}^{+\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{1}^{+\infty} = 1$

Converges, by comparison.

Problem 4: Consider the function

$$f(x) = x^4 .$$

(a) Let M be the average value of f on $[0, 3]$. Compute M.

(b) Find a value of c in $[0, 3]$ such that $f(c) = M$.

(c) The average value of a function g over $[0, x]$ is equal to x^2 for all x. Determine $g(x)$.

a) $M = \frac{1}{3-0} \int_0^3 x^4 \, dx = \frac{1}{3} \left[\frac{x^5}{5}\right]_0^3 = \frac{3^4}{5}$

b) $f(c) = M$: $c^4 = \frac{3^4}{5} \rightsquigarrow c = \frac{3}{\sqrt[4]{5}}$

c) $\frac{1}{x-0} \int_0^x g(t)\,dt = x^2 \rightsquigarrow \int_0^x g(t)\,dt = x^3$

$\xrightarrow[\frac{d}{dx}]{\text{FTC}} \underline{g(x) = 3x^2}$

Problem 5: Let

$$f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x\ .$$

Find the *arc length* of the curve $y = f(x)$ over the interval $[1, e]$.

$$\text{Arc length} = \int_1^e \sqrt{1+(f'(x))^2}\,dx = \int_1^e \sqrt{1+\left(\frac{1}{2}x - \frac{1}{2}\frac{1}{x}\right)^2}\,dx$$

$$= \int_1^e \sqrt{1+\frac{1}{4}\left(x^2+\frac{1}{x^2}-2\right)}\,dx = \int_1^e \sqrt{\frac{1}{4}\left(x^2+\frac{1}{x^2}+2\right)}\,dx$$

$$= \frac{1}{2}\int_1^e \sqrt{\left(x+\frac{1}{x}\right)^2}\,dx = \frac{1}{2}\int_1^e \left(x+\frac{1}{x}\right)dx = \frac{1}{2}\left[\frac{x^2}{2} + \ln|x|\right]_1^e$$

$$= \frac{1}{2}\left(\frac{e^2}{2} + 1 - \frac{1}{2} - 0\right) = \frac{e^2+1}{4}$$

Problem 6: Let

$$f(x) = e^{-\sqrt{24}\,x} .$$

Find N such that M_N approximates the integral

$$\int_0^{10} f(x)dx$$

with an error of at most 10^{-3}.

Hint: Here M_N denotes the N^{th} *midpoint approximation* to the integral. We have the error bound formula:

$$\left| \int_a^b f(x)dx - M_N \right| \leq K \frac{(b-a)^3}{24N^2} ,$$

where K is any real number such that $|f''(x)| \leq K$ for all $a \leq x \leq b$.

$f'(x) = -\sqrt{24}\, e^{-\sqrt{24}\,x}$

$f''(x) = 24\, e^{-\sqrt{24}\,x}$. f'' is decreasing on $[0,10]$, So the max of $|f''(x)|$ is at $x=0$: $|f''(x)| \leq \underbrace{24}_{K}$

It suffices to have $K \dfrac{(b-a)^3}{24N^2} \leqslant 10^{-3}$

$\rightsquigarrow 24 \dfrac{(10-0)^3}{24N^2} \leqslant 10^{-3}$

$\rightarrow N^2 \geqslant 10^6$

$\rightsquigarrow \boxed{N \geqslant 1000}$