1. *(2 points per part)* Each of the following multiple choice problems has one correct answer. Circle it. You do not need to show any reasoning.

(a) Suppose $|\mathbf{a} \times \mathbf{b}| > -\mathbf{a} \cdot \mathbf{b} > 0$. Then the angle between $\mathbf{a}$ and $\mathbf{b}$ is between...

(i) $0°$ and $45°$. (ii) $45°$ and $90°$. $\boxed{\text{(iii) } 90° \text{ and } 135°.}$ (iv) $135°$ and $180°$.

If $-\vec{a}\cdot\vec{b} > 0$, then $\vec{a}\cdot\vec{b} < 0$, so $\theta$ is obtuse

$|\vec{a}||\vec{b}|\sin\theta > -|\vec{a}||\vec{b}|\cos\theta$, so $\tan(\theta) < -1$

(b) Suppose $\text{proj}_{\mathbf{a}}\mathbf{b} = \langle 1, -1, 1 \rangle$. Then $\mathbf{b}$ could be...

(i) $\langle 2, -2, 2 \rangle$. (ii) $\langle -1, 1, -1 \rangle$. (iii) $\langle 2, 2, 2 \rangle$. $\boxed{\text{(iv) } \langle 2, 3, 4 \rangle.}$

$\text{proj}_{\langle 1,-1,1\rangle}\langle x,y,z\rangle = \frac{x-y+z}{3}\langle 1,-1,1\rangle = \langle 1,-1,1\rangle$, so $x-y+z=3$

(c) The intersection of the hyperboloid $x^2 + y^2 - z^2 = 1$ and the $xy$-plane is...

(i) a line. $\boxed{\text{(ii) a circle.}}$ (iii) a hyperbola. (iv) the empty set.

$x^2+y^2-0^2=1$

(d) The surface $z = f(x, y) = x^3 + y^3 - 3x - 3y$ has a local maximum of...

(i) $f(1, 1)$. (ii) $f(1, -1)$. (iii) $f(-1, 1)$. $\boxed{\text{(iv) } f(-1, -1).}$

$f_x(x,y) = 3x^2-3$

$f_y(x,y) = 3y^2-3$

$\left.\begin{array}{l} f_{xx}(x,y) = 6x \\ f_{yy}(x,y) = 6y \end{array}\right\}$ should both be negative

$f_{xy}(x,y) = 0$

(e) A lamina occupies the disc $x^2 + y^2 \leq 1$, and the density at $(x, y)$ is $\rho(x, y) = x^3 + y^2 + 2$. The center of mass of the lamina is...

(i) at the origin. $\boxed{\text{(ii) on the } x\text{-axis.}}$ (iii) on the $y$-axis. (iv) none of these.

symmetric about both axes

$\rho(x,y) = \rho(x,-y)$

But $\rho(x,y) > \rho(-x,y)$

2. *(4 points per part)* For each part, consider the space curve of the vector function

$$\mathbf{r}(t) = \langle t^2 + 1, \cos(t) + 4t, 3t \rangle.$$

(a) Find parametric equations for the line tangent to the space curve at $t = 0$.

$\vec{r}(0) = \langle 1, 1, 0 \rangle$

$\vec{r}'(0) = \langle 2t, -\sin(t) + 4, 3 \rangle$

$\vec{r}'(0) = \langle 0, 4, 3 \rangle$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 1 + 4t \\ z &= 3t \end{aligned}}$$

(b) Find the unit tangent vector to the space curve at $t = 0$.

$$\vec{T}(0) = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{\langle 0, 4, 3 \rangle}{\sqrt{4^2 + 3^2}} = \boxed{\left\langle 0, \frac{4}{5}, \frac{3}{5} \right\rangle}$$

(c) Find the curvature of the space curve at $t = 0$.

$\vec{r}''(t) = \langle 2, -\cos(t), 0 \rangle$

$\vec{r}''(0) = \langle 2, -1, 0 \rangle$

$\vec{r}'(0) \times \vec{r}''(0) = \langle 0, 4, 3 \rangle \times \langle 2, -1, 0 \rangle = \langle 3, 6, -8 \rangle$

$$K = \frac{|\langle 3, 6, -8 \rangle|}{|\langle 0, 4, 3 \rangle|^3} = \boxed{\frac{\sqrt{109}}{5^3}}$$

3. *(6 points per part)* For parts (a) and (b), let $\mathcal{S}$ be the implicitly defined surface

$$x\cos(z) + y^2 z - x^2 e^y + 20 = z.$$

(a) Find $\dfrac{\partial z}{\partial x}$ for points on $\mathcal{S}$.

$\frac{\partial}{\partial x}$

$$\cos(z) - x\sin(z)\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial x} - 2xe^y = \frac{\partial z}{\partial x}$$

$$\cos(z) - 2xe^y = \frac{\partial z}{\partial x}\left(1 + x\sin(z) - y^2\right)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\cos(z) - 2xe^y}{1 + x\sin(z) - y^2}}$$

(b) Find all intersections of $\mathcal{S}$ with the $\underbrace{x\text{-axis}}_{y=0,\ z=0}$.

$$x - x^2 + 20 = 0$$

$$x^2 - x - 20 = 0$$

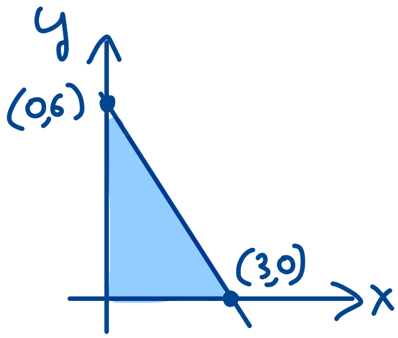
$$(x-5)(x+4) = 0$$

$$x = 5 \text{ or } -4$$

$$\boxed{(5,0,0)\ \&\ (-4,0,0)}$$

4. *(14 points)* Let $\mathcal{D}$ be the triangular region with vertices $(0,0)$, $(0,6)$, and $(3,0)$.

Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy - 2x$ on $\mathcal{D}$.




Critical points:

$f_x(x,y) = 2x + y - 2 = 0 \rightarrow y = 2$

$f_y(x,y) = x = 0$

$(0,2)$ is a crit. pt

Boundary

Left $x=0$ $(0 \le y \le 6)$

$f(0,y) = 0$

No special points

Bottom: $y=0$ $(0 \le x \le 3)$

$f(x,0) = x^2 - 2x$

$f'(x) = 2x - 2 = 0 \rightarrow$ check $(1,0)$

Top right: $y = 6-2x$ $(0 \le x \le 3)$

$f(x, 6-2x) = x^2 + x(6-2x) - 2x$

$= -x^2 + 4x$

$f'(x) = -2x + 4 = 0$

$x = 2,\ y = 6 - 2(2) = 2$

$\hookrightarrow$ check $(2,2)$

Points to check

crit pt: $f(0,2) = 0$

vertices:
$f(0,6) = 0$
$f(0,0) = 0$
$f(3,0) = 3$

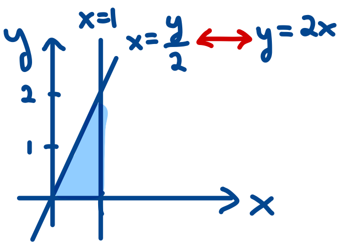
crit #s on boundary:
$f(1,0) = \boxed{-1}$ ← min
$f(2,2) = \boxed{4}$ ← max

5. *(6 points per part)* **The two parts of this problem are unrelated.**

   (a) Evaluate the iterated integral

$$\int_0^2 \int_{\frac{y}{2}}^1 y\cos(x^3-1)dx\,dy.$$

Reverse order




$$\int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx = \int_0^1 \left(\frac{1}{2}y^2\cos(x^3-1)\right)\Bigg]_{y=0}^{y=2x} dx$$

$$= \int_0^1 2x^2\cos(x^3-1)\,dx = \int_{-1}^{0} \frac{2}{3}\cos(u)\,du$$

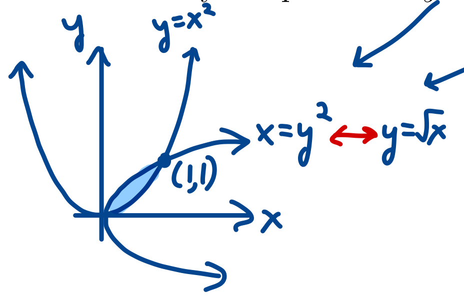
$u = x^3-1$

$du = 3x^2\,dx$

$$= \frac{2}{3}\sin(u)\Bigg]_{-1}^{0} = -\frac{2}{3}\sin(-1) = \boxed{\frac{2}{3}\sin(1)}$$

   (b) Find the volume of the solid under the plane $3x+2y-z=0$ and above the region enclosed by the parabolas $y=x^2$ and $x=y^2$.

$\underbrace{3x+2y-z=0}$ → $z = 3x+2y$



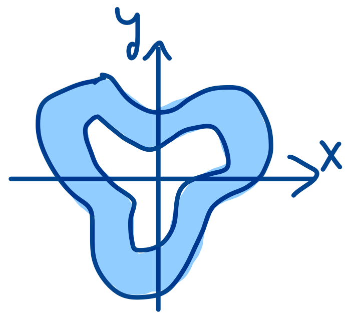

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (3x+2y)\,dy\,dx$$

$$= \int_0^1 \left(3xy+y^2\right)\Bigg]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(3x^{3/2}+x-3x^3-x^4\right)dx = \left(\frac{6}{5}x^{5/2}+\frac{1}{2}x^2-\frac{3}{4}x^4-\frac{1}{5}x^5\right)\Bigg]_0^1$$

$$= \frac{6}{5}+\frac{1}{2}-\frac{3}{4}-\frac{1}{5} = \boxed{\frac{3}{4}}$$

6. *(12 points)* Set up and evaluate a double integral in polar coordinates to calculate the area of the region between the two polar curves $r = 6 + 2\sin(3\theta)$ and $r = 3 + 2\sin(3\theta)$.




$$\text{Area} = \int_0^{2\pi} \int_{3+2\sin(3\theta)}^{6+2\sin(3\theta)} 1\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} r^2\right)\Big]_{r=3+2\sin(3\theta)}^{r=6+2\sin(3\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\left(6+2\sin(3\theta)\right)^2 - \left(3+2\sin(3\theta)\right)^2\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(36 + 24\sin(3\theta) + 4\sin^2(3\theta) - 9 - 12\sin(3\theta) - 4\sin^2(3\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(27 + 12\sin(3\theta)\right) d\theta = \frac{1}{2}\left(27\theta - 4\cos(3\theta)\right)\Big]_0^{2\pi}$$

$$= \frac{1}{2}\left(54\pi - 4 + 4\right) = \boxed{27\pi}$$

7. *(7 points per part)* Let $f(x) = e^{2x} + \ln(1-x) - x^2$.

(a) Find the second Taylor polynomial, $T_2(x)$, for $f(x)$ based at $b = 0$.

$f(x) = e^{2x} + \ln(1-x) - x^2 \qquad f(0) = 1$

$f'(x) = 2e^{2x} - \frac{1}{1-x} - 2x \qquad f'(0) = 1$

$f''(x) = 4e^{2x} - \frac{1}{(1-x)^2} - 2 \qquad f''(0) = 1$

$$\boxed{T_2(x) = 1 + x + \frac{1}{2}x^2}$$

(b) Find (and justify) an error bound for $|f(x) - T_2(x)|$ on the interval $[-0.5, 0.5]$.

$f'''(x) = \underbrace{8e^{2x}}_{\leq 8e} - \underbrace{\frac{2}{(1-x)^3}}_{\leq 16}$

So $|f'''(x)| \leq 8e + 16$ on $[-0.5, 0.5]$

$M = 8e + 16$ works

(other answers are possible, but $f'''(\frac{1}{2})$ is <u>not</u> a valid $M$.)

$$|T_2(x) - f(x)| \leq \frac{1}{6}(8e+16)\left(\frac{1}{2}\right)^3 = \boxed{\frac{e+2}{6}}$$

8. For this problem, you may use the following basic Taylor series:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

(a) *(6 points)* Find the Taylor series for $g(x) = \dfrac{1}{4+9x^2}$ based at $b = 0$.

Give your answer **using $\Sigma$-notation** and **list the first three nonzero terms**.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$\downarrow \div 4$

$$\frac{1}{4-4x} = \sum_{k=0}^{\infty} \frac{x^k}{4}$$

$\downarrow x \to \frac{-9x^2}{4}$

$$\frac{1}{4+9x^2} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 9^k x^{2k}}{4^{k+1}}} = \boxed{\frac{1}{4} - \frac{9}{16}x^2 + \frac{81}{64}x^4} + \dots$$

(b) *(5 points)* Find the Taylor series for $h(x) = \arctan\left(\dfrac{3x}{2}\right)$ based at $b = 0$.

Give your answer **using $\Sigma$-notation** and **list the first three nonzero terms**.

$$h'(x) = \frac{\frac{3}{2}}{1+\frac{9x^2}{4}} = \frac{6}{4+9x^2}$$

$$\frac{6}{4+9x^2} = \sum_{k=0}^{\infty} \frac{6(-1)^k 9^k x^{2k}}{4^{k+1}}$$

$\downarrow \int dx$

$$\arctan\left(\frac{3}{2}x\right) = \boxed{\sum_{k=0}^{\infty} \frac{6(-1)^k 9^k x^{2k+1}}{4^{k+1}(2k+1)}} = \boxed{\frac{6}{4}x - \frac{54}{48}x^3 + \frac{486}{320}x^5} + \dots$$

(c) *(3 points)* Find the open interval of convergence for the series you found in part (b).

$\frac{1}{1-x}$ converges for $-1 < x < 1$

$\downarrow \div 4$

no change

$\downarrow x \to \frac{-9x^2}{4}$

$\frac{1}{4+9x^2}$ converges for $-1 < \frac{-9x^2}{4} < 1$

$\downarrow$ mult. by 6, integrate

no change

$$-1 < \frac{-9x^2}{4} < 1$$

$$\underbrace{-\frac{4}{9} < x^2}_{\text{always}} \underbrace{< \frac{4}{9}}$$

$$-\frac{2}{3} < x < \frac{2}{3}.$$

So $\boxed{\left(-\frac{2}{3}, \frac{2}{3}\right)}$