Math 126, Section C, Autumn 2012, Solutions to Midterm I

1. Answer the following question regarding the picture below

We know \( \vec{AC} = \langle 2, 6, 2 \rangle, \vec{BD} = \langle 4, 0, -2 \rangle \) and \( A = (0, 2, -1) \).

(a) (4 points) Compute the two vectors \( \vec{u} = \vec{AB} = \vec{DC} \) and \( \vec{v} = \vec{AD} = \vec{BC} \). From \( \vec{v} + \vec{u} = \vec{AC} = \langle 2, 6, 2 \rangle \) and \( \vec{v} - \vec{u} = \vec{BD} = \langle -2, 0, -2 \rangle \) we have \( 2\vec{v} = \langle 6, 0, 0 \rangle \) and \( \vec{v} = \langle 3, 0, 0 \rangle \) and \( \vec{u} = \langle 0, 0, 0 \rangle \).

(b) (3 points) Find the coordinates of the points \( B \) and \( C \). From \( \langle 2, 6, 2 \rangle = \vec{AC} \) we get \( C = (2, 8, 1) \). From \( \langle -1, 3, 2 \rangle = \vec{AB} \) we get \( B = (-1, 5, 1) \).

(c) (3 points) The line containing \( B \) and \( E \) is perpendicular to the line containing \( A \) to \( D \) as shown in the picture. Find the coordinates of the point \( E \).

\[
\vec{AE} = \text{proj}_u \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v}
\]

\[
= \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} + \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

so \( E = (1, 3, -1) \).

2. Given two planes

\[
P_1 : \quad 2x - y + z = 5
\]

and

\[
P_2 : \quad 3x + 2y - z = 3.
\]

(a) (6 points) Find parametric equations for the line of intersection of the two planes. Check that your line is indeed on both planes.

The direction vector for the line is \( \vec{n}_1 \times \vec{n}_2 = \langle 2, -1, 1 \rangle \times \langle 3, 2, 1 \rangle = \langle -1, 5, 7 \rangle \).

To compute a point on both planes we can take \( x = 0 \) and solve \( 0 - y + z = 5 \) and \( 0 + 2y - z = 3 \) to get \( y = 8 \) and \( z = 13 \) so the point is \( (0, 8, 13) \).

Therefore the line equations are

\[
x = 0 - t, \quad y = 8 + 5t, \quad z = 13 + 7t.
\]

(b) (3 points) Find the equation of a third plane \( P_3 \) which contains that line and the point \( P(0, 7, 2) \).

The normal for the plane is \( \vec{n} = \vec{n}_3 \times \vec{PQ} \) where \( Q = (0, 8, 13) \). So the normal vector is \( \langle -1, 5, 7 \rangle \times \langle 0, 1, 1 \rangle = \langle 48, 11, -1 \rangle \). Therefore, the plane equation is

\[
48x + 11y - z = 75.
\]

(c) (1 point) Find the line of intersection of the planes \( P_1 \) and \( P_3 \).

The same as the line in part (a)

3. Answer the following.

(a) (6 points) Match the following vector functions with the curves they trace in space. The positive z-axis points up in the graphs. Write the letter of the graph next to the corresponding vector function.

\[
\begin{align*}
\vec{r}_1(t) &= \langle t + 3, 2t - 1, -t + 4 \rangle \quad &\vec{r}_2(t) &= \langle 2t + 3, 2t - 1, t + 4 \rangle \quad &\vec{r}_3(t) &= \langle t \cos(t), t, \frac{t \sin(t)}{2} \rangle \quad &D \\
\vec{r}_4(t) &= \langle t, \sin(t), 0 \rangle \quad &\vec{r}_5(t) &= \langle t + 1, 2t^2 - 5t + 1, t^3 \rangle \quad &\vec{r}_6(t) &= \langle \cos(t), 10t, \frac{\sin(t)}{2} \rangle \quad &F
\end{align*}
\]
(b) (4 points) Find the vector equation of the tangent line to \( \mathbf{r}(t) = (t + 1, 2t^2 - 5t + 1, t^3) \) at the point where \( t = 2 \).

Plugging in \( t = 2 \), the point is \((2 + 1, 8 - 10 + 1, 8) = (3, -1, 8)\). The direction vector is \( \mathbf{r}'(2) \).

We compute \( \mathbf{r}'(t) = \langle 1, 4t - 5, 3t^2 \rangle \) so \( \mathbf{r}'(2) = \langle 1, 3, 12 \rangle \). Therefore the line equation is \( \mathbf{r}(t) = \langle 3 + t, -1 + 3t, 8 + 12t \rangle \).

4. Given the equation 

\[ x^2 - 4y^2 + 4z^2 + 8y = 4, \]

(a) (5 points) Identify the surface and sketch it. Label your axes so I can see the orientation. Label any points you think are important, for example, if you have a sphere, label its center.

Completing the square and rearranging terms we get

\[ \frac{x^2}{4} + z^2 = (y - 1)^2 \]

which is a double cone with the \( y \) axis running through it. The two cones meet at the point \((0, 1, 0)\).

(b) (4 points) Find the point(s) of intersection of the above surface and the line given by

\[ x = 8t \quad y = 5t + 1 \quad z = 3 - t. \]

We solve

\[ (8t)^2 - 4(5t + 1 - 1)^2 + 4(3 - t)^2 = 0 \]

simplifying we get

\[ -t^2 - 6t + 9 = 0 \]

so \( t = -3/2 \) or \( t = -3/4 \), giving us the points \((-12, -13/2, 9/2)\) and \((-6, -11/4, 15/4)\).

(c) (1 point) Write one vector function which gives a curve on this cone. There are many answers to this question.

Here any three functions \( f(t), g(t), h(t) \) with

\[ \frac{f(t)^2}{4} + h(t)^2 = (g(t) - 1)^2 \]

would give a curve on the cone. For example \( \mathbf{r}(t) = \langle 2t, t+1, 0 \rangle \) or \( \mathbf{r}(t) = \langle 2t \cos t, t+1, t \sin t \rangle \).