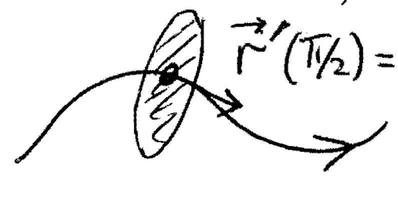
1 (12 points total) Consider the curve given by the vector equation

$$\vec{r}(t) = \langle \sin(2t), t, \cos(2t) \rangle.$$

(a) (6 points) Verify that the point $P(0, \pi/2, -1)$ lies on the curve, and find the equation of the **normal plane** to the curve at P.

$t = \pi/2$, $\sin(2t) = \sin(\pi) = 0$, $\cos(2t) = \cos(\pi) = -1$,

So Yes!



$\vec{r}\,'(\pi/2) = \vec{n}$

$\vec{r}\,'(t) = \langle 2\cos(2t), 1, -2\sin(2t) \rangle$

$\vec{r}\,'(\pi/2) = \langle 2\cos(\pi), 1, -2\sin(\pi) \rangle = \langle -2, 1, 0 \rangle$

normal plane : $-2(x-0) + 1(y - \frac{\pi}{2}) + 0(z+1) = 0$

$$\boxed{-2x + y = \pi/2}$$

(b) (6 points) Find the curvature of the curve at all points (it may depend on t).

$\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)| / |\vec{r}\,'(t)|^3$

$\vec{r}\,'(t) = \langle 2\cos(2t), 1, -2\sin(2t) \rangle$, $|\vec{r}\,'(t)| = \sqrt{4\cos^2(2t) + 1 + 4\sin^2(2t)}$

$\vec{r}\,''(t) = \langle -4\sin(2t), 0, -4\cos(2t) \rangle$

$= \sqrt{5}$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \langle -4\cos(2t), \underbrace{8\cos^2(2t) + 8\sin^2(2t)}_{8}, 4\sin(2t) \rangle$

$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{16\cos^2(2t) + 64 + 16\sin^2(2t)} = \sqrt{80} = 4\sqrt{5}$

$\kappa(t) = 4\sqrt{5} / \sqrt{5}^3 = \boxed{4/5}$

2 (12 points total)

(a) (6 points) Verify that the point $P(1, 2, 2)$ lies on the surface $z = f(x, y) = \sqrt{1 - x^3 + y^2}$, and find the equation of the tangent plane to this surface at P.

$$x_0 = 1, \quad y_0 = 2, \quad z_0 = \sqrt{1 - x_0^3 + y_0^2} = \sqrt{1 - 1 + 4} = 2$$

so Yes!

$$f_x(x,y) = \frac{-3x^2}{2\sqrt{1-x^3+y^2}}, \quad f_x(1,2) = \frac{-3}{2 \cdot 2} = -\frac{3}{4}$$

$$f_y(x,y) = \frac{2y}{2\sqrt{1-x^3+y^2}}, \quad f_y(1,2) = \frac{4}{4} = 1$$

$$z - 2 = -\frac{3}{4}(x-1) + 1(y-2) \text{ or } \boxed{\frac{3}{4}x - y + z = \frac{3}{4}}$$

(b) (6 points) Use linear approximation for $f(x, y)$ based at $(1, 2, 2)$ to estimate the number $\sqrt{1 - (0.96)^3 + (2.02)^2}$.

$$f(0.96, 2.02) \approx f(1,2) + f_x(1,2)(0.96 - 1) + f_y(1,2)(2.02 - 2)$$

$$= 2 - \frac{3}{4}(-0.04) + 1(0.02)$$

$$= 2 + 0.03 + 0.02 = \boxed{2.05}$$

3 (12 points) Find the points of local maximum and minimum and saddle points for the function

$$f(x,y) = x^2 + 2xy^2 - 4xy$$

$f_x = 2x + 2y^2 - 4y = 0$

$f_y = 4xy - 4x = 0 \Rightarrow 4x(y-1) = 0 \Rightarrow x = 0$ or $y = 1$

Case 1: $x = 0$, $2y^2 - 4y = 0 \Rightarrow 2y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$

get $(0,0), (0,2)$

Case 2: $y = 1$, $2x + 2 - 4 = 0 \Rightarrow x = 1$, get $(1,1)$

Critical points: $\boxed{(0,0), (0,2), (1,1)}$

2nd derivatives test: $f_{xx} = 2$, $f_{yy} = 4x$, $f_{xy} = 4y - 4$

$$D = \begin{vmatrix} 2 & 4y-4 \\ 4y-4 & 4x \end{vmatrix}$$

$D(0,0) = \begin{vmatrix} 2 & -4 \\ -4 & 0 \end{vmatrix} < 0 \Rightarrow \boxed{(0,0) \text{ is a saddle point}}$

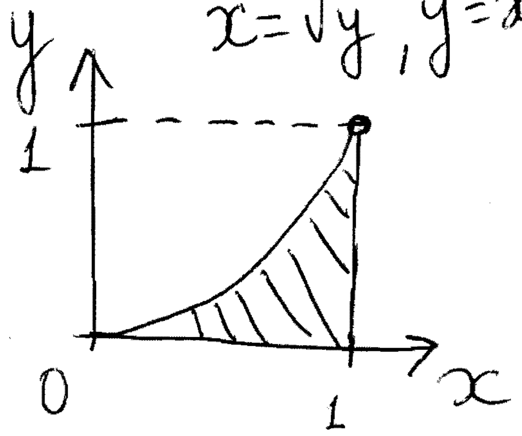
$D(0,2) = \begin{vmatrix} 2 & 4 \\ 4 & 0 \end{vmatrix} < 0 \Rightarrow \boxed{(0,2) \text{ is a saddle point}}$

$D(1,1) = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} > 0$, $f_{xx}(1,1) > 0 \Rightarrow \boxed{(1,1) \text{ is a local min}}$

4 (14 points total)

(a) (7 points) Evaluate the following integral (you may want to reverse the order of integration):

$x=\sqrt{y},\ y=x^2$



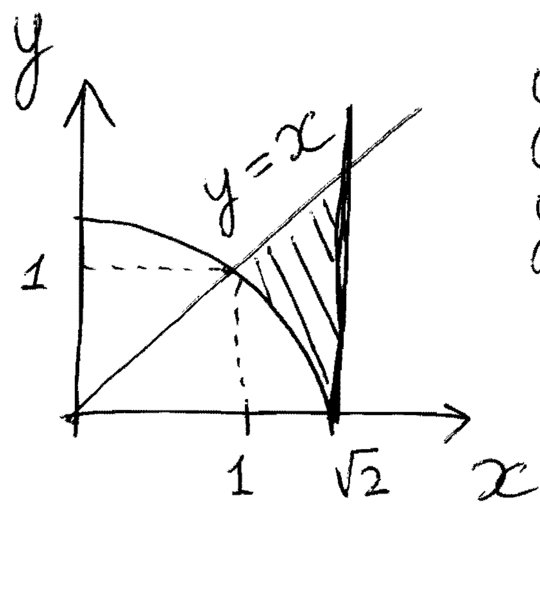
$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1\int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx$$

$$= \int_0^1 \frac{e^{x^2}}{x^3}\,\frac{y^2}{2}\Big|_{y=0}^{x^2}\,dx$$

$$= \int_0^1 \frac{e^{x^2}}{x^3}\,\frac{x^4}{2}\,dx = \int_0^1 \frac{xe^{x^2}}{2}\,dx = \frac{e^{x^2}}{4}\Big|_0^1 = \boxed{\frac{e-1}{4}}$$

(b) (7 points) Convert the following integral to polar coordinates **(do not attempt to evaluate!)**:

$$\int_1^{\sqrt{2}} \int_{\sqrt{2-x^2}}^{x} f(x,y)\,dy\,dx = \int_0^{\pi/4}\int_{\sqrt{2}}^{\sqrt{2}/\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



$y=\sqrt{2-x^2}$

$y^2+x^2=2$

$r^2=2,\ \boxed{r=\sqrt{2}}$

$x=\sqrt{2}$

$r\cos\theta=\sqrt{2}$

$\underline{r=\sqrt{2}/\cos\theta}$