1. **[8 points]** An ant is standing on the surface $z = x^3 - 3xy + e^{xy}$ at the point $(1, 0)$.

(a) **[4 points]** If the ant were to walk East (that is, in the positive x direction), would it move up or down? Explain your reasoning.

$$\frac{\partial z}{\partial x} = 3x^2 - 3y + y\, e^{xy}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,0)} = 3 > 0$$

Since the slope in the x-direction is positive the ant would move up.

(b) **[4 points]** Use differentials to estimate the ant's change in altitude when the ant travels from $(1, 0)$ to $(0.95, 0.12)$.

$$\frac{\partial z}{\partial x} = 3x^2 - 3y + y\, e^{xy} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,0)} = 3$$

$$\frac{\partial z}{\partial y} = -3x + x\, e^{xy} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(1,0)} = -2$$

$$dz = \left.\frac{\partial z}{\partial x}\right|_{(1,0)} dx + \left.\frac{\partial z}{\partial y}\right|_{(1,0)} dy$$

$$= 3(0.95 - 1) + (-2)(0.12 - 0)$$

$$= 3(-0.05) - 2(0.12)$$

$$= \boxed{-0.39} \cong \text{change in altitude.}$$

2. [**12 points**] Consider the function:

$$f(x, y) = xy^2 - 2x + 2$$

(a) Find and classify each of its critical points as a local minimum, local maximum, or saddle point.

$$\begin{cases} f_x(x,y) = y^2 - 2 = 0 & \Leftrightarrow y = \pm\sqrt{2} \\ f_y(x,y) = 2xy = 0 & \Leftrightarrow x = 0 \text{ or } \cancel{y = 0} \end{cases}$$

← can not have this because $f_x \neq 0$ when $y=0$

We have 2 critical points: $\boxed{(0, \pm\sqrt{2})}$

$$\left.\begin{array}{l} f_{xx}(x,y) = 0 \\ f_{xy}(x,y) = f_{yx}(x,y) = 2y \\ f_{yy}(x,y) = 2x \end{array}\right\} \Rightarrow D(x,y) = \begin{vmatrix} 0 & 2y \\ 2y & 2x \end{vmatrix} = -4y^2$$

Since $D(0, \pm\sqrt{2}) = -4(\pm\sqrt{2})^2 = -8 < 0$

$\boxed{\text{both critical points are SADDLE POINTS}}$

(b) Find the abolute maximimum value of this function on the region $D = \{(x, y)|x^2 + y^2 \leq 1\}$.

1) no critical points inside the region

2) Boundary: $x^2+y^2=1 \Rightarrow \boxed{y^2 = 1-x^2}$

$g(x) := f(x, \pm\sqrt{1-x^2}) = x(1-x^2) - 2x - 2 = -x^3 - x + 2$

$g'(x) = -3x^2 - 1$ ← never zero $\Rightarrow$ no CP's on the boundary.

3) Endpoints: The domain of $y^2 = 1-x^2$ is $-1 \leq x \leq 1$

So the endpoints are
$x = -1 \Rightarrow y = 0 \Rightarrow \boxed{(-1, 0)}$
$x = 1 \Rightarrow y = 0 \Rightarrow \boxed{(1, 0)}$.

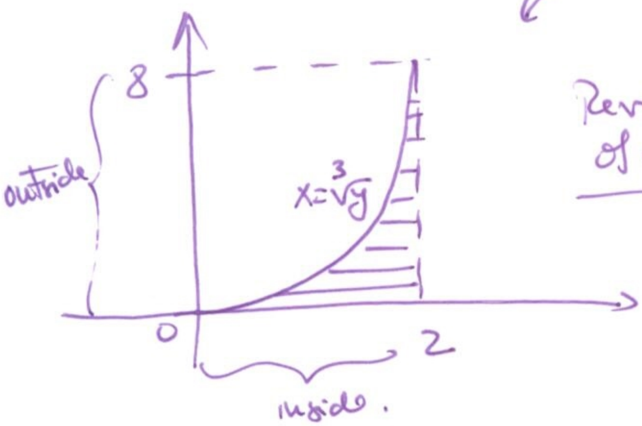
$f(-1,0) = 4$

$f(1,0) = 0$

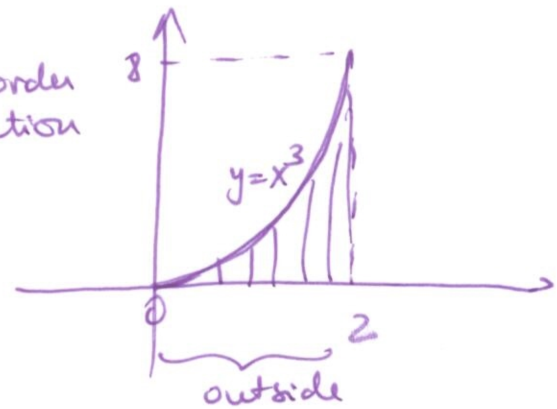
$\boxed{\text{Hence max. value is } z = 4}$. at $(x,y) = (-1, 0)$

3. [**8 points**] Evaluate:

$$\int_0^8 \int_{y^{\frac{1}{3}}}^2 \frac{y^2 e^{x^2}}{x^8} dx dy$$



Reverse order of integration $\longrightarrow$



$$\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y^2 e^{x^2}}{x^8} dx\, dy = \int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8} dy\, dx$$

$$= \int_0^2 \frac{e^{x^2}}{x^8} \left(\frac{y^3}{3}\right)\Big|_0^{x^3} dx$$

$$= \frac{1}{3}\int_0^2 \frac{e^{x^2}}{x^8} (x^9 - 0)\, dx$$

$$= \frac{1}{3} \int_0^2 x e^{x^2} dx \qquad \boxed{\begin{array}{c} u = x^2 \\ \frac{1}{2}du = x dx \end{array}}$$

$$= \frac{1}{3}\int_0^4 \frac{1}{2} e^u\, du$$

$$= \frac{1}{6}\, e^u \Big|_0^4 = \boxed{\frac{1}{6}(e^4 - 1)}$$

4. [**10 points**]

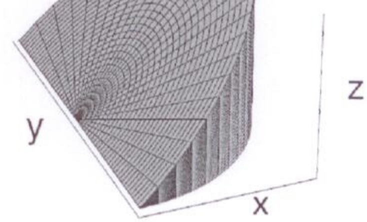
(a) Find the volume of the wedge shaped solid that lies above the xy-plane, below the plane $z = x$, and within the solid cylinder $x^2 + y^2 \leq 9$.

$$V = \int_{-\pi/2}^{\pi/2} \int_0^3 \overbrace{r\cos\theta}^{x}\, r\,dr\,d\theta$$

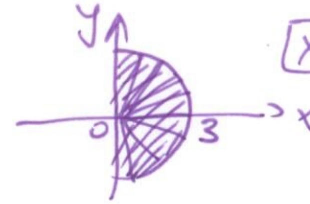
$$= \int_{-\pi/2}^{\pi/2} \cos\theta \left(\frac{r^3}{3}\right)\Big|_0^3 d\theta$$

$$= 2\int_0^{\pi/2} 9\cos\theta\, d\theta$$

$$= 18\,(\sin\theta)\Big|_0^{\pi/2} = \boxed{18}.$$



Use Polar Coord.

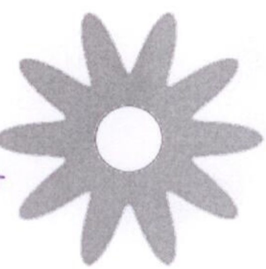


$\boxed{x = r\cos\theta}$.

(b) Find the area of the flower-like region which is given in polar coordinates (r, θ) as

$$1 \leq r \leq 3 + \cos(10\theta)$$

The picture of this region can be admired to the right.



$$A = \int_0^{2\pi} \int_1^{3+\cos(10\theta)} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2}\right)\Big|_1^{3+\cos(10\theta)} d\theta = \frac{1}{2}\int_0^{2\pi} (3+\cos(10\theta))^2 - 1\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 8 + 6\cos(10\theta) + \underbrace{\cos^2(10\theta)}\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 8 + 6\cos(10\theta) + \frac{1+\cos(20\theta)}{2}\, d\theta.$$

$$= \frac{1}{2}\left(8\theta + 6\frac{\sin(10\theta)}{10} + \frac{1}{2}\theta + \frac{\sin(20\theta)}{40}\right)\Big|_0^{2\pi}.$$

$$= \frac{1}{2}\left(8(2\pi) + \frac{1}{2}(2\pi)\right) = \boxed{\frac{17\pi}{2}}.$$

5. [**12 points**] A point on the outer rim of a badly thrown frisbee moves on a curve $\mathbf{r}(t)$, with **acceleration**:

acceleration: $\mathbf{r}''(t) =< 0, -\cos(t), -\sin(t) >$

We know that $\mathbf{r}'(0) =< 1, 0, 1 >$ and $\mathbf{r}(0) =< 0, 1, 0 >$.

(a) [**3 points**] Find $\mathbf{r}(t)$.

Velocity: $\left.\begin{array}{l}\vec{r}'(t) = \langle C_1, -\sin t + C_2, \cos t + C_3 \rangle \\ \vec{r}'(0) = \langle 1,0,1 \rangle = \langle C_1, C_2, 1 + C_3 \rangle\end{array}\right\} \Rightarrow \vec{r}'(t) = \langle 1, -\sin t, \cos t \rangle$

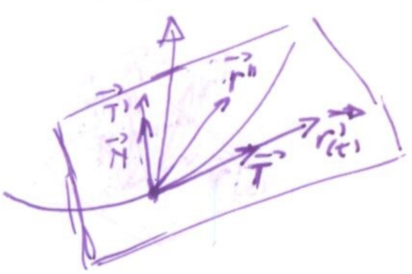
position: $\left.\begin{array}{l}\vec{r}(t) = \langle t + K_1, \cos t + K_2, \sin t + K_3 \rangle \\ \vec{r}(0) = \langle 0,1,0 \rangle = \langle K_1, 1 + K_2, K_3 \rangle\end{array}\right\} \Rightarrow \boxed{\vec{r}(t) = \langle t, \cos t, \sin t \rangle}$

(b) [**3 points**] Find the arclength of the curve from $t = 0$ to $t = 2\pi$.

$$L = \int_0^{2\pi} \| \vec{r}'(t) \| dt = \int_0^{2\pi} \sqrt{1^2 + (-\sin t)^2 + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\pi}$$

(c) [**6 points**] Find the equation of the osculating plane at $t = \frac{\pi}{2}$.



The osculating plane contains the vectors $\vec{N}$ & $\vec{T}$ so we can take $\vec{B} = \vec{T} \times \vec{N}$ as the normal vector. But it also contains $\vec{r}'$ ($\parallel \vec{T}$) and $\vec{r}''$ (we proved this when we decomposed acceleration into normal & tangential components) so it's easier to take

the normal vector $\vec{n} = \vec{r}'(\pi/2) \times \vec{r}''(\frac{\pi}{2})$.

$$= \langle 1, -1, 0 \rangle \times \langle 0, 0, -1 \rangle = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$\Rightarrow \boxed{\vec{n} = \langle 1, 1, 0 \rangle}$$

The point P at $t = \pi/2$ is given by $\boxed{\vec{r}(\pi/2) = \langle \pi/2, 0, 1 \rangle}$

The plane equation is:

$$1(x - \pi/2) + 1(y - 0) + 0(z - 1) = 0$$

which simplifies to $\boxed{x + y = \frac{\pi}{2}}$.