1. (12 pts) Let $z = f(x, y)$ be a function defined by the implicit equation

$$xz + e^z = y^2$$

(a) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$xz + e^z = y^2$$

product rule $\quad \downarrow \frac{\partial}{\partial x} \quad$ chain rule

$$1 \cdot z + x\frac{\partial z}{\partial x} + e^z \frac{\partial z}{\partial x} = 0$$

$$(x + e^z)\frac{\partial z}{\partial x} = -z$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-z}{x + e^z}}$$

$$xz + e^z = y^2$$

chain $\quad \downarrow \frac{\partial}{\partial y}$

$$x \cdot \frac{\partial z}{\partial y} + e^z \frac{\partial z}{\partial y} = 2y$$

$$(x + e^z)\frac{\partial z}{\partial y} = 2y$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{2y}{x + e^z}}$$

(b) Use your answer in (a) to give a vector tangent to the trace of the surface $z = f(x, y)$ on the plane $x = 2$ at the point $(2, 1, 0)$. (The question did not ask for the tangent line, just give a tangent vector.)

tangent slope to the trace on $x=2$ is $\frac{\partial z}{\partial y}(2,1,0)$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,1,0)} \overset{(a)}{=} \left.\frac{2y}{x+e^z}\right|_{(2,1,0)} = \frac{2 \times 1}{2 + e^0} = \frac{2}{3}$$

tangent vector $\vec{v} = \langle 0, 1, \frac{\partial z}{\partial y}(2,1,0) \rangle = \boxed{\langle 0, 1, \frac{2}{3} \rangle}$

2. (15 pts) Consider the function $f(x, y) = x^2 + \frac{1}{3}y^3 - 3y$.

(a) Find all the critical points of f in $\mathbb{R}^2$, then determine if f has a local maximum, local minimum, or a saddle point at each of the critical point. (No need to find the f value at the critical points for this part.)

$$\left.\begin{array}{l} f_x = 2x = 0 \Rightarrow x = 0 \\ f_y = \frac{1}{3}(3y^2) - 3 = y^2 - 3 = 0 \Rightarrow y = \pm\sqrt{3} \end{array}\right\} \Rightarrow \boxed{\text{cr pts: } (0, \sqrt{3}),\ (0, -\sqrt{3})}$$

$$f_{xx} = 2,\quad f_{xy} = 0 \Rightarrow D = f_{xx}f_{yy} - f_{xy}^2 = 2(2y) = 4y$$
$$f_{yy} = 2y,$$

	$D = 4y$	$f_{xx} = 2$
$(0, \sqrt{3})$	$4\sqrt{3} > 0$	$2 > 0$
$(0, -\sqrt{3})$	$4(-\sqrt{3}) < 0$	

$\Rightarrow$ local min @ $(0, \sqrt{3})$

$\Rightarrow$ saddle pt @ $(0, -\sqrt{3})$

(b) Find the absolute maximum and absolute minimum value of f on the disk $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$.

The critical pts $(0, \pm\sqrt{3})$ found in (a) are not in D.

On the boundary circle $x^2 + y^2 = 1$, $x^2 = 1 - y^2$

substitute

$$f(x, y) = x^2 + \frac{1}{3}y^3 - 3y$$
$$\Rightarrow g(y) = 1 - y^2 + \frac{1}{3}y^3 - 3y, \quad -1 \leq y \leq 1$$

$$g'(y) = -2y + y^2 - 3 = (y-3)(y+1) = 0$$
$$\Rightarrow \text{cr pt: } \cancel{y = 3} \text{ or } y = -1$$

when $y = -1$, $x = 0$, $f(0, -1) = 0 + \frac{1}{3}(-1)^3 - 3(-1)$
$$= -\frac{1}{3} + 3 = \frac{8}{3}$$

At the other end pt $y = 1$, $x = 0$, $f(0, 1) = -\frac{8}{3}$

$$\boxed{\begin{array}{ll} \text{abs. max}: & f = \frac{8}{3} \\ \text{abs. min}: & f = -\frac{8}{3} \end{array}}$$

3. (12 pts) Consider the integral

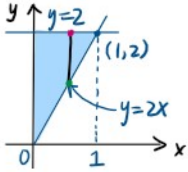
$$\int_0^1 \int_{2x}^2 \cos(y^2)\, dy dx$$

(a) Sketch and shade the region of integration on the xy plane.

region D (type I)

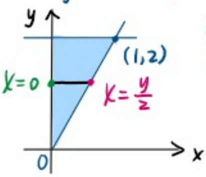
$2x \le y \le 2$

$0 \le x \le 1$



(b) Reverse the order of integration and evaluate the integral.

region D (type II)



$0 \le x \le \frac{y}{2}$

$0 \le y \le 2$

$$\int_0^2 \int_0^{\frac{y}{2}} \cos(y^2)\, dx dy$$

$$= \int_0^2 \left(\cos(y^2) x \Big|_{x=0}^{x=\frac{y}{2}} \right) dy$$

$$= \int_0^2 \cos(y^2) \frac{y}{2}\, dy$$

u-sub

$u = y^2$

$du = 2y\,dy$

$\int \cos(u) \frac{1}{4} du = \frac{1}{4} \sin(u)$

$$= \frac{1}{4} \sin(y^2) \Big|_{y=0}^{y=2}$$

$$= \frac{1}{4} \sin(4) - 0$$

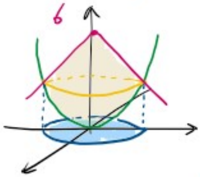
4. (10 pts) Setup a double integral in **Polar coordinate** to find the volume of the solid below the cone $z = 6 - \sqrt{x^2 - y^2}$ and above the paraboloid $z = x^2 + y^2$. Do **NOT** evaluate the integral.

$$\int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{2}} \boxed{((6-r) - r^2)\, r}\, d\boxed{r}\, d\boxed{\theta}$$

To get full and/or partial credit, show work to arrive the upper/lower limits of the integral.

top: cone $z = 6 - \sqrt{x^2+y^2} \rightsquigarrow z = 6 - r \ (r > 0)$

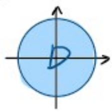
bottom: paraboloid $z = x^2 + y^2 \rightsquigarrow z = r^2$



intersection: $6 - r = r^2$

$r^2 + r - 6 = (r-2)(r+3) = 0$

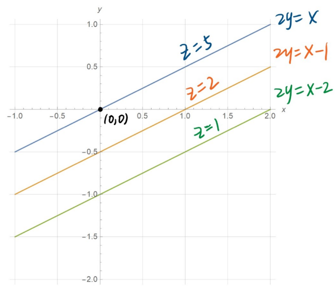
$\Rightarrow r = 2$ or $\cancel{r = -3}$



$D: 0 \le r \le 2$

$0 \le \theta \le 2\pi$

5. (11 pts) The graph below illustrated 3 level curves of an unknown function $z = f(x, y)$. The lines are: $2y = x$ with $z = 5$; $2y = x - 1$ with $z = 2$; $2y = x - 2$ with $z = 1$.



(a) Give a reasonable (and as precise as possible) estimation for $f_x(0, 0)$ and $f_y(0, 0)$.

$$f_x(0,0) \approx \frac{\Delta z}{\Delta x} = \frac{f(1,0)-f(0,0)}{1-0} = \frac{2-5}{1} = -3$$

$$f_y(0,0) \approx \frac{\Delta z}{\Delta y} = \frac{f(0,-0.5)-f(0,0)}{-0.5-0} = \frac{2-5}{-0.5} = 6$$

(b) Based on the given level curves, would you say the second derivative $f_{xx}(0, 0)$ is positive or negative? Briefly explain your answer.

$f_{xx} > 0$, b/c $f_x(0,0) \approx -3$

$$f_x(1,0) \approx \frac{\Delta z}{\Delta x} = \frac{f(2,0)-f(1,0)}{2-1} = \frac{1-2}{1} = -1 \quad \left(\text{or the average: } \frac{LHS+RHS}{2} = \frac{-3+(-1)}{2} = -2\right)$$

$$\Rightarrow f_{xx}(0,0) \approx \frac{f_x(1,0)-f_x(0,0)}{1-0} \approx \frac{-1-(-3)}{1} > 0 \quad (f_x \text{ increases when } x \text{ increases})$$

(c) (**This is a challenge question that worth 3 points, do not spend much time on this question unless you have confidently done the other problems.**)

Find a function $z = f(x, y)$ that has the given three level curves. The answer is not unique.

Note that the level curves are $2y - x = k$, $k = 0, -1, -2$

let $z = f(x,y)$ be of the form: $z = F(2y-x)$ with $\begin{cases} 5 = F(0) \\ 2 = F(-1) \\ 1 = F(-2) \end{cases}$

For example, $F(t) = at^2 + bt + c$ (quadratic function)

$$\begin{cases} F(0) = a(0)+b(0)+c = 5 & \Rightarrow c = 5 \\ F(-1) = a(-1)^2+b(-1)+c = 2 & \Rightarrow a-b+c=2 \\ F(-2) = a(-2)^2+b(-2)+c = 1 & \Rightarrow 4a-2b+c=1 \end{cases} \quad \Rightarrow a=1,\ b=4,\ c=5.$$

$z = a(2y-x)^2 + b(2y-x) + c \rightsquigarrow z = 1(2y-x)^2 + 4(2y-x) + 5$ works.