1 (16 points) Evaluate the following double integrals.

(a) (8 points) $\iint_R x\sec^2(xy)\,dA, \quad R = [0, \pi/4] \times [0, 1]$

$$\begin{aligned}
\int_0^{\pi/4}\int_0^1 x\sec^2(xy)\,dy\,dx &= \int_0^{\pi/4} \tan(xy)\Big|_{y=0}^{1}\,dx \\
&= \int_0^{\pi/4} \tan(x)\,dx \\
&= \ln(\sec(x))\Big|_0^{\pi/4} \\
&= \ln\sqrt{2}
\end{aligned}$$

(b) (8 points) $\iint_D 2xy\,dA$ D is the triangle with vertices $(0, 0)$, $(1, 2)$ and $(0, 3)$.

$$\begin{aligned}
\int_0^1\int_{2x}^{3-x} 2xy\,dy\,dx &= \int_0^1 xy^2\Big|_{2x}^{3-x}\,dx \\
&= \int_0^1 9x - 6x^2 - 3x^3\,dx \\
&= \frac{9}{2}x^2 - 2x^3 - \frac{3}{4}x^4\Big|_0^1 \\
&= \frac{7}{4}
\end{aligned}$$

2 (9 points) Find the absolute maximum of the function $f(x, y) = x + 2y - xy$ on the closed rectangular region with vertices $(0, 0)$, $(0, 2)$, $(3, 0)$ and $(3, 2)$.

First find the critical points.

$f_x(x, y) = 1 - y = 0$ *gives* $y = 1$
$f_y(x, y) = 2 - x = 0$ *gives* $x = 2$.

There is one critical point at $(2, 1)$. $f(2, 1) = 2$.

Now find the maximum of $f(x, y)$ *on the boundary. There are four parts to this.*

I. $x = 0$, $0 \leq y \leq 2$
$f(0, y) = 2y$ *is increasing so the maximum is* $f(0, 2) = 4$.

II. $y = 0$, $0 \leq x \leq 3$
$f(x, 0) = x$ *is increasing so the maximum is* $f(3, 0) = 3$.

III. $x = 3$, $0 \leq y \leq 2$
$f(3, y) = 3 - y$ *is decreasing so the maximum is* $f(3, 0) = 3$.

IV. $y = 2$, $0 \leq x \leq 3$
$f(x, 2) = 4 - x$ *is decreasing so the maximum is* $f(0, 2) = 4$.

Thus the maximum value of the function on this rectangle is $f(0, 2) = 4$.

3 (8 points) If three resistors with resistances R_1, R_2 and R_3 are connected in parallel, then the total resistance R of the circuit is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Suppose that the resistances are measured in ohms with $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, and that there is a possible error of 0.5 ohms in each measurement. Use differentials to estimate the maximum error in the calculated value of R.

$$\frac{1}{R} = \frac{1}{25} + \frac{1}{40} + \frac{1}{50} \quad \textit{so} \quad R = \frac{200}{17}.$$

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2} \qquad -\left(\frac{17}{200}\right)^2 \cdot \frac{\partial R}{\partial R_1} = -\frac{1}{25^2} \qquad \frac{\partial R}{\partial R_1} = \frac{64}{289}$$

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_2} = -\frac{1}{R_2^2} \qquad -\left(\frac{17}{200}\right)^2 \cdot \frac{\partial R}{\partial R_2} = -\frac{1}{40^2} \qquad \frac{\partial R}{\partial R_2} = \frac{25}{289}$$

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_3} = -\frac{1}{R_3^2} \qquad -\left(\frac{17}{200}\right)^2 \cdot \frac{\partial R}{\partial R_3} = -\frac{1}{50^2} \qquad \frac{\partial R}{\partial R_3} = \frac{16}{289}$$

$$\begin{aligned} dR &= \frac{\partial R}{\partial R_1} \cdot dR_1 + \frac{\partial R}{\partial R_2} \cdot dR_2 + \frac{\partial R}{\partial R_3} \cdot dR_3 \\ &= \frac{64}{289} \cdot 0.5 + \frac{25}{289} \cdot 0.5 + \frac{16}{289} \cdot 0.5 \\ &= \frac{105}{578} \\ &\approx 0.18 \end{aligned}$$

4 (8 points) Find all the points on the curve $r = 1 + \cos\theta$ where the tangent line is horizontal.

We wish to find the points where $\frac{dy}{d\theta} = 0$

$y = r\sin\theta = (1+\cos\theta)(\sin\theta) = \sin\theta + \cos\theta\sin\theta$

$$\begin{aligned} \frac{dx}{d\theta} &= \cos\theta - \sin^2\theta + \cos^2\theta \\ &= \cos\theta - (1-\cos^2\theta) + \cos^2\theta \\ &= 2\cos^2\theta + \cos\theta - 1 \\ &= (2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Thus $\cos\theta = -1, \frac{1}{2}$ and $\theta = \pi, \pm\frac{\pi}{3}$.
In polar coordinates, the points are $(0,\pi)$, $(3/2, \pi/3)$ and $(3/2, -\pi/3)$

5 (9 points) Let $\mathbf{r}(t) = 3t^2\,\mathbf{i} + t^3\,\mathbf{j} + 3t^2\,\mathbf{k}$. Find all times t when the normal component of acceleration is equal to 8.

We must solve $\dfrac{|r' \times r''|}{|r'|} = 8$ *or* $|r' \times r''| = 8\,|r'|$ *for* t.

$r'(t) = \langle 6t, 3t^3, 6t \rangle$
$r''(t) = \langle 6, 6t, 6 \rangle$
$r' \times r'' = \langle -18t^2, 0, 18t^2 \rangle = 18t^2 \langle -1, 0, 1 \rangle$
$|r' \times r''| = 18t^2\sqrt{2}$

$|r'(t)| = \sqrt{(6t)^2 + (3t^2)^2 + (6t)^2} = 3t\sqrt{t^2+8}$

$18t^2\sqrt{2} = 8 \cdot 3t\sqrt{t^2+8}$ *(note that at* $t = 0$ *we have* $a_N = 0$*.)*
$3\sqrt{2}\,t = 4\sqrt{t^2+8}$
$18t^2 = 16(t^2+8)$
$t^2 = 64$

The solutions are $t = -8, 8$