1 (16 points) Evaluate the following double integrals.

(a) (8 points) $\displaystyle\iint_R \frac{x}{1+xy}\,dA, \quad R=[0,1]\times[0,2]$

$$\begin{aligned}
\iint_R \frac{x}{1+xy}\,dA &= \int_0^1\int_0^2 \frac{x}{1+xy}\,dy\,dx \\
&= \int_0^1 \ln(1+xy)\Big|_0^2\,dx \\
&= \int_0^1 \ln(1+2x)\,dx \qquad (u=\ln(1+2x),\ dv=dx) \\
&= x\ln(1+2x)\Big|_0^1 - \int_0^1 \frac{2x}{1+2x}\,dx \\
&= \ln 3 - [x-\frac{1}{2}\ln(1+2x)]\Big|_0^1 \\
&= -1+\frac{3}{2}\ln 3
\end{aligned}$$

(b) (8 points) $\displaystyle\iint_D xy^2\,dA,$ D is the triangle with vertices $(0,0)$, $(0,2)$ and $(1,2)$.

$$\begin{aligned}
\iint_D xy^2\,dA &= \int_0^2\int_0^{y/2} xy^2\,dx\,dy \\
&= \int_0^2 \frac{1}{2}x^2y^2\Big|_{x=0}^{y/2}\,dy \\
&= \int_0^2 \frac{1}{8}y^4\,dy \\
&= \frac{4}{5}
\end{aligned}$$

2 (9 points) Let $f(x,y) = x^2 - y^2 + 4\ln(xy)$. Find all points on the surface where the tangent plane is parallel to the plane $6x = 2y + z$.

The normal vector to $6x = 2y + z$ *is* $\langle 6, -2, -1 \rangle$.

The normal vector to the tangent plane to $z = f(x,y)$ *at the point* (x,y) *is* $\langle f_x(x,y), f_y(x,y), -1 \rangle$.

Thus we need to solve the equations $f_x(x,y) = 6$ *and* $f_y(x,y) = -2$ *for* x *and* y.

$f_x(x,y) = 2x + \dfrac{4}{x}$ *and* $f_y(x,y) = -2y + \dfrac{4}{y}$

$$\begin{aligned} -2y + \frac{4}{y} &= -2 \\ 0 &= y^2 - y - 2 \\ &= (y-2)(y+1) \end{aligned}$$

$$\begin{aligned} 2x + \frac{4}{x} &= 6 \\ 0 &= x^2 - 3x + 2 \\ &= (x-1)(x-2) \end{aligned}$$

We get 4 points: $(2,2)$, $(1,2)$, $(2,-1)$ *and* $(1,-1)$. *But the second 2 are not in the domain of* $f(x,y)$.

$f(2,2) = 4\ln 4$ *and* $f(1,2) = -3 + 4\ln 2$

The points are $(1, 2, -3 + 4\ln 2)$ *and* $(2, 2, 4\ln 4)$.

3 (8 points) Compute the equation of the tangent line to the curve $r = 1 + 2\sin\theta$ at the point where $\theta = \pi/6$. Give your answer in exact form.

We need the slope and the xy-coordinates of the point.

At $\theta = \pi/6$ *we have* $r = 2$*. Thus* $x = r\cos\theta = \sqrt{3}$ *and* $y = r\sin\theta = 1$.

$$\begin{aligned} x &= r\cos\theta \\ &= (1 + 2\sin\theta)\cos\theta \\ \frac{dx}{d\theta} &= 2\cos^2\theta - (1 + 2\sin\theta)\sin\theta \\ &= \frac{1}{2} \qquad \textit{when you plug in } \theta = \pi/6 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= (1 + 2\sin\theta)\sin\theta \\ \frac{dy}{d\theta} &= 2\cos\theta\sin\theta + (1 + 2\sin\theta)\cos\theta \\ &= \frac{3\sqrt{3}}{2} \qquad \textit{when you plug in } \theta = \pi/6 \end{aligned}$$

The slope is $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = 3\sqrt{3}$.

The equation of the line is $y - 1 = 3\sqrt{3}\left(x - \sqrt{3}\right)$.

4 (8 points) Let $\mathbf{r}(t) = 3t\,\mathbf{i} + 3t^2\,\mathbf{j} + 2t^3\,\mathbf{k}$. Calculate the curvature at the time $t = -2$.

We use the formula $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$.

$$\begin{aligned} \mathbf{r}'(t) &= \langle 3, 6t, 6t^2 \rangle \\ \mathbf{r}'(-2) &= \langle 3, -12, 24 \rangle = 3\,\langle 1, -4, 8 \rangle \\ \mathbf{r}''(t) &= \langle 0, 6, 12t \rangle \\ \mathbf{r}''(-2) &= \langle 0, 6, -24 \rangle = 6\,\langle 0, 1, -4 \rangle \\ \mathbf{r}'(-2) \times \mathbf{r}''(-2) &= 18\,\langle 8, 4, 1 \rangle \\ \kappa &= \frac{2}{243} \end{aligned}$$

5 (9 points) Find the absolute maximum of the function $f(x,y) = (2x-1)\cos\left(\frac{\pi}{2}\,y\right)$ on the closed rectangular region with vertices $(0,1)$, $(0,4)$, $(3,1)$ and $(3,4)$.

First find the critical points in the region.
$f_x(x,y) = 2\cos\left(\frac{\pi}{2}\,y\right) = 0$ *gives* $y = 1, 3$.
$f_y(x,y) = -\frac{\pi}{2}(2x-1)\sin\left(\frac{\pi}{2}\,y\right) = 0$ *gives* $y = 2, 4$ *or* $x = \frac{1}{2}$.
The critical points are $(\frac{1}{2}, 1)$ *and* $(\frac{1}{2}, 3)$. *Note that* $f(x,y) = 0$ *at both of these points.*

Next consider the four line segments that make up the boundary.
i) $x = 0$, $1 \le y \le 4$: *Here* $f(0,y) = -\cos\left(\frac{\pi}{2}\,y\right)$. *This has a maximum of 1 at* $(0,3)$.
ii) $x = 3$, $1 \le y \le 4$: *Here* $f(3,y) = 5\cos\left(\frac{\pi}{2}\,y\right)$. *This has a maximum of 5 at* $(3,4)$.
iii) $y = 1$, $0 \le x \le 3$: *Here* $f(x,1) = 0$, *a constant. This has a maximum of 0 everywhere.*
iv) $y = 4$, $0 \le x \le 3$: *Here* $f(x,4) = 2x-1$. *This has a maximum of 5 at* $(3,4)$.

Thus 5 is the maximum value of $f(x,y)$ *on the rectangular region.*