Math 126 A - Spring 2019
Midterm Exam Number Two Solutions
May 21, 2019

Name: ______________________________ Student ID no. : ____________________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- This exam consists of **five** problems on **four** double-sided pages.
- Show all work for full credit.
- You may use a TI-30X IIS calculator during this exam. Other calculators and electronic devices are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- Draw a box around your final answer to each problem.
- **Do not write within 1 centimeter of the edge!** Your exam will be scanned for grading.
- If you run out of room, write on the back of the first or last page and indicate that you have done so. If you still need more room, raise your hand and ask for an extra page.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

1. **[10 points]** Let $f(x,y) = \cos x + \cos y$. Find all the critical points (x,y) of f with

$$-\pi/2 \leq x \leq \pi/2 \text{ and } -\pi/2 \leq y \leq \pi/2$$

and classify them (*local max*, *local min*, or *saddle point*) using the second derivative test.

We need to find all points (x_0, y_0) satisfying

$$-\pi/2 \leq x_0 \leq \pi/2 \text{ and } -\pi/2 \leq y_0 \leq \pi/2$$

such that the partial derivatives of f satisfy

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

We have

$$\begin{aligned} f_x(x,y) &= -\sin x \\ &\implies x_0 = 0 \\ f_y(x,y) &= -\sin y \\ &\implies y_0 = 0 \end{aligned}$$

Here, we know that we must have $x_0 = y_0 = 0$ since these are the only values of x and y within the bounds of consideration; this gives our only critical point $(0,0)$. We now use the second derivative test to determine the nature of this point:

$$f_{xx}(x,y) = -\cos x, \quad f_{yy}(x,y) = -\cos y, \quad f_{xy}(x,y) = 0$$

Thus, plugging into our formula:

$$\begin{aligned} D(0,0) &= f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2 \\ &= -\cos(0) \cdot (-\cos(0)) - 0 \\ &= 1 > 0 \end{aligned}$$

Thus this point is either a local minimum or a local maximum. Since $f_{xx}(0,0) = -1 < 0$, the function has a local maximum at this point.

2. **[10 points]** Find the critical points of the function $f(x, y) = x^4 + y^4 - 4xy$ and classify them (*local max*, *local min*, or *saddle point*) using the second derivative test.

We seek points (x_0, y_0) such that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

$$\begin{aligned} f_x(x, y) &= 4x^3 - 4y \\ &\implies y_0 = x_0^3 \\ f_y(x, y) &= 4y^3 - 4x \\ &\implies x_0 = y_0^3 \end{aligned}$$

Together, these conclusions yield $x_0 = x_0^9$ and $y_0 = y_0^9$, hence $x_0 = 0, 1,$ or -1 and $y_0 = 0, 1,$ or -1. Since $x_0 = y_0^3$ and $y_0 = x_0^3$, this gives only three possibilities: $(0, 0), (1, 1), (-1, -1)$. We use the second derivative test to classify these critical points:

$$f_{xx}(x, y) = 12x^2, \quad f_{yy}(x, y) = 12y^2, \quad f_{xy}(x, y) = 4$$

Thus

$$\begin{aligned} D(0, 0) &= -16 < 0 \\ D(1, 1) &= 144 - 16 > 0 \\ D(-1, -1) &= 144 - 16 > 0 \end{aligned}$$

Thus $(0, 0)$ is a saddle point, and $(1, 1)$ and $(-1, -1)$ are local minima or maxima. To see which, we note that

$$f_{xx}(1, 1) = 12 > 0, \quad f_{xx}(-1, -1) = 12 > 0$$

Thus, f has local mininma at $(1, 1)$ and $(-1, -1)$.

3. **[10 points]** Compute the double integral

$$\int_R xye^{xy^2}\,dA$$

where $R = [0,1] \times [0,2]$ is the rectangle $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}$.

$$\iint_R xye^{xy^2}\,dA = \int_0^1 \int_0^2 xye^{xy^2}\,dydx.$$

To evaluate the inner integral, we use the u-substitution $u = xy^2$, so $du = 2xydy$, or $\frac{1}{2}du = xydy$. Then the limits of integration change to $x(0)^2 = 0$ to $x(2)^2 = 4x$, so we have

$$\iint_R xye^{xy^2}\,dA = \int_0^1 \int_0^{4x} \frac{1}{2}e^u\,dudx = \int_0^1 \frac{1}{2}\left(e^u\Big|_0^{4x}\right)dx = \int_0^1 \frac{1}{2}\left(e^{4x} - 1\right)dx.$$

Now

$$\int_0^1 \frac{1}{2}\left(e^{4x} - 1\right)dx = \frac{1}{2}\left(\frac{1}{4}e^{4x}\Big|_0^1 - x\Big|_0^1\right) = \frac{1}{2}\left(\left(\frac{1}{4}e^4 - \frac{1}{4}\right) - (1-0)\right) = \frac{1}{8}\left(e^4 - 5\right).$$

4. **[10 points]** Compute the volume of the solid above the xy-plane, below the parabaloid $z = x^2 + y^2$, and inside the cylinder $x^2 + y^2 = 1$.

   This volume can be represented by

$$\int\int_R x^2 + y^2 dA.$$

   Where R is the region $\{x^2 + y^2 \leq 1\}$, that is, the unit disk centered at the origin. In polar coordinates

$$R = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\},$$

   and $f(x, y) = x^2 + y^2 = r^2$, and dA is $rdrd\theta$. So our integral can be rewritten as

$$\int_0^{2\pi}\int_0^1 r^3 drd\theta.$$

   Computing, we get,

$$\begin{aligned}
\int_0^{2\pi}\int_0^1 r^3 drd\theta &= \int_0^{2\pi} \left(\frac{1}{4}r^4\Big|_0^1\right) d\theta \\
&= \int_0^{2\pi} \frac{1}{4} d\theta \\
&= \frac{1}{4}\theta\Big|_0^{2\pi} \\
&= \frac{1}{4}(2\pi - 0) \\
&= \frac{\pi}{2}.
\end{aligned}$$

5. **[10 points]** Find the absolute minima and maxima of the function $x^2 - y^2$ on the region $R = \{x^2 + y^2 \leq 1\}$.

Let $f(x, y) = x^2 - y^2$. We have

$$\begin{aligned} f_x(x,y) =& 2x; \\ f_y(x,y) =& -2y. \end{aligned}$$

We first look for critical points in the interior. The function has derivatives which are defined everywhere, so we need to look for points (x, y) where

$$f_x(x, y) = f_y(x, y) = 0.$$

Using our computation of the partials above, we see that $(0, 0)$ is the only critical point. $f_{xx} = 2$, $f_{yy} = -2$, and $f_{xy} = f_{yx} = 0$. Hence $D(x, y) = -4$. By Second Derivative Test, $(0, 0)$ is a saddle point.

On the boundary, we have $x^2 + y^2 = 1$, which we can rewrite as $y^2 = 1 - x^2$. So $y = \pm\sqrt{1 - x^2}$. So our function on the boundary can be thought of as two functions, one on top of the circle

$$g^+(x) = f(x, +\sqrt{1 - x^2}) = x^2 - (1 - x^2) = 2x^2 - 1$$

and the other on the bottom of the circle

$$g^-(x) = f(x, -\sqrt{1 - x^2}) = x^2 - (1 - x^2) = 2x^2 - 1,$$

both defined on $[-1, 1]$. Note that since $g^+ = g^-$, we can just consider the function $g(x) = 2x^2 - 1$. We have $g'(x) = 4x$, which has a critical point $x = 0$, and we have to also evaluate it at the endpoints 1 and -1. We have

$$\begin{aligned} g(-1) =& 1; \\ g(0) =& -1; \\ g(1) =& 1. \end{aligned}$$

We conclude that the global maximum is 1 (achieved at $(1, 0)$ and $(-1, 0)$) and the global minimum is -1 (achieved at $(0, 1)$ and $(0, -1)$).