Math 126, Section C - Winter 2015
Midterm II
March 3, 2015

Name: ____

Student ID Number: ________________

Section: CA 11:30-12:20 by Sam ☐ CB 12:30-1:20 by Sam ☐

CC 11:30-12:20 by Ru-Yu ☐ CD 12:30-1:20 by Ru-Yu ☐

exercise	**possible**	**score**
1	12	
2	14	
3	12	
4	12	
total	50	

- Check that this booklet has all the exercises indicated above.
- TURN OFF YOUR CELL PHONE.
- Write your name and your student ID.
- This is a 50 minute test.
- You may use a scientific calculator and one 8.5×11 inch sheet of (two-sided) handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- Unless otherwise indicated, your answers should be exact instead of decimal approximations. For example $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.78.
- Unless otherwise indicated, show your work and justify all your answers.
- Do not use scratch paper! Do your scratch work at the bottom of the page or on the back side of the preceding sheet.

Sample solution

Exercise 1 (12 points).

The equation

$$xyz + 2x^2y^2 - 3y^2z^3 = -2$$

determines a surface in $\mathbb{R}^3$. Find the tangent plane to the surface at the point $(1,2,1)$.

Hint: Use implicit differentiation.

Solution: We consider $z(x,y)$ as a function of x,y. Then around the point $(1,2,1)$, the function $z(x,y)$ is determined by the equation

$$xy \cdot z(x,y) + 2x^2y^2 - 3y^2z(x,y)^3 = -2.$$

Taking partial derivatives on both sides of the equation, we obtain

$$yz + xyz_x + 4xy^2 - 9y^2z^2z_x \overset{!}{=} 0 \Rightarrow yz + 4xy^2 = (9y^2z^2 - xy)z_x$$
$$\Rightarrow \quad z_x = \frac{yz + 4xy^2}{9y^2z^2 - xy} \overset{x=1,y=2,z=1}{=} \frac{9}{17}$$

and

$$xz + xyz_y + 4x^2y - 6yz^3 - 9y^2z^2z_y \overset{!}{=} 0 \Rightarrow xz + 4x^2y - 6yz^3 = (9y^2z^2 - xy)z_y$$
$$\Rightarrow \quad z_y = \frac{xz + 4x^2y - 6yz^3}{9y^2z^2 - xy} \overset{x=1,y=2,z=1}{=} -\frac{3}{34}$$

The equation of the plane is then

$$\boxed{z = \frac{9}{17}(x-1) - \frac{3}{34} \cdot (y-2) + 1}$$

Exercise 2 (10+2+2=14 points).

Consider the function

$$f(x,y) = \frac{1}{x^2+1} - xy + y$$

a) Determine and classify all critical points of f.

b) We want to understand better, how the function f behaves for points (x,y) that have a large magnitude.

i) Find a vector function $\vec{r}(t) = (x(t),y(t))$ so that $f(x(t),y(t)) \to \infty$ as $t \to \infty$.

ii) Find a vector function $\vec{r}(t) = (x(t),y(t))$ so that $f(x(t),y(t)) \to -\infty$ as $t \to \infty$.

Solution: For a). We have

$$f_x = -\frac{2x}{(x^2+1)^2} - y \qquad\qquad f_y = -x+1$$

Then we solve

$$\begin{bmatrix} f_x = 0 \\ f_y = 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -\frac{2x}{(x^2+1)^2} - y = 0 \\ -x+1 = 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -\frac{2}{(1+1)^2} - y = 0 \\ x = 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y = -\frac{1}{2} \\ x = 1 \end{bmatrix}$$

Hence the only critical point is $(x,y) = (1,-\frac{1}{2})$. The Hessian matrix is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} \frac{8x^2}{(x^2+1)^3} - \frac{2}{(x^2+1)^2} & -1 \\ -1 & 0 \end{pmatrix}$$

Plugging in $(x,y) = (1,-\frac{1}{2})$ we get

$$\begin{vmatrix} f_{xx}(1,-\frac{1}{2}) & f_{xy}(1,-\frac{1}{2}) \\ f_{xy}(1,-\frac{1}{2}) & f_{yy}(1,-\frac{1}{2}) \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$

Hence $\boxed{(1,-\frac{1}{2}) \text{ is a saddle point}}$.

b) i) For example for $\vec{r}(t) = (0,t)$ the function value goes to ∞ for $t \to \infty$ and to $-\infty$ for $t \to -\infty$.

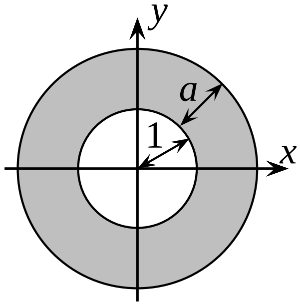
ii) For example for $\vec{r}(t) = (t,1)$ [or $\vec{r}(t) = (0,-t)$], the function tends to $-\infty$ as $t \to \infty$.

Exercise 3 (12 points).

We want to construct a lamina in form of a ring that has inner radius 1 and thickness a. If the density of the lamina is given by

$$\rho(x,y) = x^4 + 2x^2y^2 + y^4,$$

then how do we need to choose a so that the mass of the lamina will be exactly $728 \cdot \frac{\pi}{3}$?



Observe that $\rho(x,y) = (x^2+y^2)^2$ (which is r^4 in polar coordinates). We integrate

$$\begin{aligned}
& \int_0^{2\pi}\int_1^{1+a} r\cdot\rho(r\cdot\sin(\theta), r\cdot\sin(\theta))\,dr\,d\theta \\
= \;& \int_0^{2\pi}\int_1^{1+a} r\cdot\Big((r\sin(\theta)^2 + (r\cos(\theta))^2\Big)^2\,dr\,d\theta \\
= \;& \int_0^{2\pi}\int_1^{1+a} r^5\cdot\underbrace{(\sin^2(\theta)+\cos^2(\theta))^2}_{=1} dr\,d\theta \\
= \;& \int_0^{2\pi}\left[\frac{1}{6}r^6\right]_{r=1}^{r=1+a} d\theta \\
= \;& \frac{1}{6}\int_0^{2\pi}((1+a)^6-1)d\theta \\
= \;& \frac{1}{6}\cdot 2\pi\cdot((1+a)^6-1)
\end{aligned}$$

This quantity is $728\cdot\frac{\pi}{3}$ exactly for $\boxed{a=2}$.

Exercise 4 (12 points).

Compute the integral

$$\int_{-4}^{0}\int_{\sqrt{-y}}^{2} y\cdot\sqrt{5+x^5}\,dx\,dy$$

Hint: Reverse the order of integration.

Solution: We reverse the order of integration and obtain

$$\begin{aligned}
\int_{-4}^{0}\int_{\sqrt{-y}}^{2} y\cdot\sqrt{5+x^5}\,dx\,dy &= \int_{0}^{2}\int_{-x^2}^{0} y\cdot\sqrt{5+x^5}\,dy\,dx\\
&= \int_{0}^{2}\left[\frac{1}{2}y^2\sqrt{5+x^5}\right]_{y=-x^2}^{0} dx\\
&= -\frac{1}{2}\int_{0}^{2} x^4\sqrt{5+x^5}\,dx\\
&= -\frac{1}{10}\int_{0}^{2} \underbrace{5x^4}_{=\frac{d}{dx}(5+x^5)}\sqrt{5+x^5}\,dx\\
&= -\frac{1}{10}\cdot\frac{2}{3}\left[(5+x^5)^{3/2}\right]_{0}^{2}\\
&= \boxed{-\tfrac{1}{15}(37^{3/2}-5^{3/2})} \approx -14.2587
\end{aligned}$$