Math 126 A - Winter 2019
Midterm Exam Number Two
February 28, 2019

Name: ______________________________ Student ID no. : ________________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- This exam consists of **five** problems on **four** double-sided pages.
- Show all work for full credit.
- You may use a TI-30X IIS calculator during this exam. Other calculators and electronic devices are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- $\boxed{\text{Draw a box}}$ around your final answer to each problem.
- **Do not write within 1 centimeter of the edge!** Your exam will be scanned for grading.
- If you run out of room, write on the back of the first or last page and indicate that you have done so. If you still need more room, raise your hand and ask for an extra page.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

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1. **[10 points total]** Consider the points $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$.

(a) **[3 points]** Find the equation for the plane passing through the points A, B, C in the form $z = a - bx - cy$.

We have $\overrightarrow{AB} = \langle -1, 1, 0\rangle$, $\overrightarrow{AC} = \langle -1, 0, 1\rangle$.The cross product of these two vectors is the normal vector

$$\overrightarrow{n} = \langle 1, 1, 1\rangle.$$

We know the plane passes through the point A, so the equation of the plane is

$$1(x - 1) + 1(y - 0) + 1(z - 0) = 0,$$

or

$$z = 1 - x - y.$$

(b) **[2 points]** Write down the function $f(x, y)$ representing the **square** of the distance from the origin $(0, 0, 0)$ to the point $(x, y, a - bx - cy)$ on the plane from part (a).

The square of the distance from the point $(x, y, 1 - x - y)$ to $(0, 0, 0)$ is given by

$$f(x, y) = (x - 0)^2 + (y - 0)^2 + (1 - x - y - 0)^2.$$

(c) **[5 points]** Find the critical points of the function $f(x, y)$ from part (b) and use the second derivative test to find the point on the plane from part (a) which is closest to the origin $(0, 0, 0)$. The partial derivative with respect to x is

$$f_x(x, y) = 2x - 2(1 - x - y) = 4x + 2y - 2.$$

The partial derivative with respect to y is

$$f_y(x, y) = 2y - 2(1 - x - y) = 4y + 2x - 2.$$

Setting both derivatives equal zero yields the system of equations

$$f_x(x, y) = 4x + 2y - 2 = 0 \text{ and } f_y(x, y) = 4y + 2x - 2 = 0.$$

Simplifying, we get

$$4x + 2y = 2 \text{ and } 4y + 2x = 2.$$

Solving this system of equations, we get

$$x = y = 1/3,$$

so $(1/3, 1/3)$ is a critical point of f. To check that this is a minimum, we apply the second derivative test. We have

$$f_{xx} = 4, f_{yy} = 4, f_{xy} = 2,$$

so

$$D(x, y) = f_{xx}f_{yy} - [f_{xy}]^2 = 12 > 0.$$

Since

$$D(1/3, 1/3) = 12 > 0 \text{ and } f_x(1/3, 1/3) = 4 > 0,$$

$(1/3, 1/3)$ is a local minimum of f by the second derivative test. Plugging $x = 1/3, y = 1/3$ into the equation of our plane, we get $z = 1 - x - y = 1/3$, so that $(1/3, 1/3, 1/3)$ is the closest point on our plane to the origin.

2. **[10 points]** Find and classify the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

The partial derivative with respect to x is

$$f_x(x, y) = 3x^2 - 3y.$$

The partial derivative with respect to y is

$$f_y(x, y) = 3y^2 - 3x.$$

To find the critical points, we set both partials equal zero.

$$f_x(x, y) = 3x^2 - 3y = 0 \text{ and } f_y(x, y) = 3y^2 - 3x = 0.$$

The critical points (after solving this system of equations) are $x = 0, y = 0$ or $x = 1, y = 1$. To classify the critical points, we apply the second derivative test:

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3,$$

so

$$D(x, y) = f_{xx}f_{yy} - [f_{xy}]^2 = 36xy - 9.$$

At our critical points, we have

at $(0, 0)$**:** $D(0, 0) = -9 < 0$, so $(0, 0)$ is a saddle point.

at $(1, 1)$**:** $D(1, 1) = 27 > 0$ and $f_{xx}(0, 0) = 6 > 0$. This critical point is a local minimum.

3. **[10 points]** Compute the double integral

$$\int_R xe^{xy} dA$$

where $R = [0,1] \times [0,2]$ is the rectangle $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}$.

$$\int_R xe^{xy} dA = \int_0^1 \int_0^2 xe^{xy} dy dx.$$

Let $u = xy$, $du = xdy$, and the limits of integration on our inner integral become 0 and $2x$. So we have

$$\int_R xe^{xy} dA = \int_0^1 \int_0^{2x} e^u du dx = \int_0^1 \left(e^u \Big|_0^{2x} \right) dx.$$

Simplifying, we get

$$\int_0^1 \left(e^{2x} - 1\right) dx = \left(\frac{e^{2x}}{2} - x \right)\Big|_0^1 = \left(\frac{e^2}{2} - 1 \right) - \left(\frac{1}{2} - 0 \right) = \frac{e^2}{2} - \frac{3}{2}.$$

4. **[10 points]** Compute the volume of the solid that lies above the region in the xy-plane bounded by the curve $y = x^2$, the horizontal line $y = 1$, and the y-axis; and under the surface $z = xy$.

Note that the function xy is positive only when x and y have the same sign. Thus, our volume is given by integrating the function xy over the right hand region bounded by the curve $y = x^2$, the horizontal line $y = 1$, and the y-axis, that is, the region in the first quadrant. We can set this integral up as

$$\int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy = \int_0^1 y \int_0^{\sqrt{y}} x\,dx\,dy = \int_0^1 y \left(\frac{x^2}{2} \Big|_0^{\sqrt{y}} \right) dy = \frac{1}{2} \int_0^1 y^2\,dy = \frac{1}{6} y^3 \Big|_0^1 = 1/6.$$

5. **[10 points total]**

(a) **[2 points]** Write down the function $f(x, y)$ that represents the **square** of the distance from the point $(11, 22, 0)$ to the point $(x, y, x^2 + y^2)$ on the paraboloid $z = x^2 + y^2$.

$$f(x, y) = (x - 11)^2 + (y - 22)^2 + (x^2 + y^2)^2.$$

(b) **[3 points]** Show that the $(1, 2)$ is a critical point of the function $f(x, y)$ from part (a).

$$f_x(x, y) = 2(x - 11) + 4x(x^2 + y^2), f_y(x, y) = 2(y - 22) + 4y(x^2 + y^2).$$

$$f_x(1, 2) = 2(1 - 11) + 4 \cdot 1 \cdot 5 = -20 + 20 = 0,$$
$$f_y(1, 2) = 2(2 - 22) + 4 \cdot 2 \cdot 5 = -40 + 40 = 0.$$

So $(1, 2)$ is a critical point.

(c) **[5 points]** Show that $(1, 2, 5)$ is the closest point on the paraboloid $z = x^2 + y^2$ to the point $(11, 22, 0)$ by showing, using the second derivative test, that $(1, 2)$ is a local minimum of the function $f(x, y)$ from part (a).

$$f_{xx}(x, y) = 2 + 12x^2 + 4y^2, f_{yy}(x, y) = 2 + 12y^2 + 4x^2, f_{xy}(x, y) = 8xy$$

Evaluating at $(1, 2)$, we get

$$f_{xx}(1, 2) = 2 + 12 + 16 = 30, f_{yy}(1, 2) = 2 + 48 + 4 = 54, f_{xy}(1, 2) = 16.$$

So

$$D(1, 2) = f_{xx}(1, 2)f_{yy}(1, 2) - (f_{xy}(1, 2))^2 = 30 \cdot 54 - (16)^2 = 1364 > 0$$

and $f_{xx}(1, 2) > 0$, so $(1, 2)$ is indeed a local minimum.