Existence and Uniqueness

In the handout on Picard iteration, we proved a local existence and uniqueness theorem for first order differential equations. The conclusion was weaker than our conclusion for first order linear differential equations because we only proved that there existed a solution on a small interval. The theorem for linear equations had a better conclusion because we found an explicit formula for a solution. For linear differential equations with order larger than 1 there is no general formula that always works. In this note, we will show how to deduce the theorem we already have for first order linear equations via Picard iteration. Then we will prove the analogous theorems for higher order equations with a similar proof.

Suppose we wish to solve the initial value problem

$$y'(t) = f(t, y(t)) \text{ with } y(t_0) = a_0. \tag{1}$$

For $a > 0$ let $R = \{(t, y) : |t - t_0| \le a,\ -\infty < y < \infty\}$.

Suppose

(i) $f(t, y)$ is continuous as a function of t, for all $(t, y) \in R$ and

(ii) there is a constant K so that for all (t, y) and (t, z) in R

$$|f(t, y) - f(t, z)| \le K|y - z|. \tag{2}$$

Note that we have not made any assumptions about the maximum of $|f|$. Let $c = \min\left(a, \frac{1}{2K}\right)$. and set

$$\mathcal{F} = \{\phi : \phi \text{ is continuous on } |t - t_0| \le a \text{ and } \phi(t_0) = a_0\}.$$

Let T be the same operator as in the proof of the Picard-Lindelöf Theorem (PLT). In this case, because R is the full strip, the operator T will automatically map $\mathcal{F}$ into $\mathcal{F}$: there is no restriction on how big $T(y)$ is. Then by the same proof, T has a fixed point, and so there is a unique solution to (1) on the interval $\{t : |t - t_0| \le c\}$. If $a \le \frac{1}{2K}$ then there is a solution on the full interval $|t - t_0| \le a$. If $a > \frac{1}{2K}$, let $t_j = t_0 + \frac{j}{2K}$, $j = 0, 1, 2, 3, \ldots$. Let y_1 be the value of the solution on $[t_0, t_1]$ at $t = t_1$. We can then solve the same equation with initial value $y(t_1) = y_1$ on the interval $[t_1, t_2]$ provided $t_2 \le t_0 + a$. The previous solution agrees with this new solution on $[t_0, t_1]$ by the uniqueness part of the PLT. Repeat the process on $[t_2, t_3]$, and after a finite number of such steps we proved the existence of a unique solution on $[t_0, a]$. For the very last step if $t_n > a$, then we can only guarantee a solution as far as $t = a$ by the theorem. Similarly we can extend to the left so that we have a solution on $t_0 - a \le t \le t_0 + a$. This proves:

Corollary. *Let $R = \{(t, y) : |t - t_0| \le a,\ -\infty < y < \infty\}$. Suppose there is a $K < \infty$ so that*

(i) $f(t, y)$ is continuous as a function of t for all $(t, y) \in R$ and

(ii) $|f(t, y) - f(t, z)| \le K|y - z|$ for all $(t, y) \in R$ and $(t, z) \in R$.

Then there exists a unique function $y(t)$ satisfying (1) for all t with $|t - t_0| \le a$.

For example the equation $y' + Ay + B = 0$ with $y(t_0) = a_0$, where A and B are continuous on a (closed) interval I, has a unique solution by the Corollary because $f(t, y) = -A(t)y - B(t)$ satisfies

$$|f(t, y) - f(t, z)| = |A(t)||y - z| \le K|y - z|,$$

where $K = \max\{|A(t)| : t \in I\}$. This is the content of our previous existence and uniqueness theorem for first order linear equations.

Similarly the non-linear equation $y' = A(t)\sin y + B(t)$, with $y(t_0) = a_0$ has a unique solution on any closed interval containing t_0 on which A and B are continuous, because

$$|\sin(y) - \sin(z)| \le \left| \int_z^y -\cos t dt \right| \le |y - z|.$$

Higher order differential equations

Next we will treat second order differential equations. Suppose we want to solve

$$y''(t) = f(t, y(t), y'(t)) \text{ with } y(t_0) = a_0 \text{ and } y'(t_0) = a_1.$$

Let $v = y'$. Then we can rewrite our equation as a coupled pair of equations:

$$y' = v \text{ and } v' = f(t, y, v) \text{ with } y(t_0) = a_0 \text{ and } v(t_0) = a_1. \tag{3}$$

This is equivalent to the pair of implicit equations

$$y(t) = a_0 + \int_{t_0}^t v(s)ds \quad \text{and} \quad v(t) = a_1 + \int_{t_0}^t f(s, y(s), v(s))ds. \tag{4}$$

Proving (3) and (4) are equivalent is just like proving the first order differential equation is equivalent to the integral equation in the proof of the PLT. Check it yourself to see if you understand why.

As in the proof of the PLT, we define an operator T which maps pairs of functions to pairs of functions given by

$$T(y,v)(t) = (a_0 + \int_{t_0}^{t} v(s)ds, \quad a_1 + \int_{t_0}^{t} f(s,y(s),v(s))ds).$$

Instead of using the absolute value to determine the distance between two numbers, we use the usual distance in the plane to measure the distance between pairs of numbers:

$$||(y,v)-(z,w)|| = \sqrt{(y-z)^2+(v-w)^2}.$$

For continuous functions y,v,z,w on an closed interval I we define

$$||(y,v)-(z,w)|| = \max_{t\in I} \sqrt{(y(t)-z(t))^2+(v(t)-w(t))^2}.$$

We suppose now that $f(t,y,v)$ is continuous as a function of t for (t,y,v) in $R = \{(t,y,v) : |t-t_0| \le a \text{ and } ||(y(t),v(t))-(a_0,a_1)|| \le b\}$. We also suppose that a Lipschitz condition holds: There is a $K<\infty$ so that for all $(t,y,v),(t,z,w)\in R$:

$$|f(t,y,v)-f(t,z,w)| \le K||(y,v)-(z,w)||. \tag{5}$$

As in the proof of PLT, set $y_0(t) = a_0$ and $v_0(t) = a_1$ for all $t \in I$, and $(y_{n+1}(t), v_{n+1}(t)) = T(y_n,v_n)(t)$. By exactly the same proof as before, the fixed point theorem gives a unique solution (y,v) to (4) on some interval J contained in I and containing the point t_0. Hence y is the unique solution to (3) on J. If the Lipschitz condition (5) holds for all $b<\infty$ then as in the first part of this note, we obtain a unique solution on all of the interval I.

Initial value second order linear differential equations can be written in the form

$$y''(t) = c(t) + a(t)y(t) + b(t)y'(t),$$

where a,b,c are continuous functions and $y(t_0) = a_0$ and $y'(t_0) = a_1$. Thus $f(t,y,v) = ay+bv+c$, and for t in a closed interval I

$$|f(t,y,v)-f(t,z,w)| \le \left(\max_{t\in I}|a(t)|\right)|y-z| + \left(\max_{t\in I}|b(t)|\right)|v-w|.$$

If $|a(t)| \le K/2$ and $|b(t)| \le K/2$ for all $t\in I$ then (5) holds because $|y-z| \le \sqrt{|y-z|^2+|v-w|^2}$ and $|v-w| \le \sqrt{|y-z|^2+|v-w|^2}$.

For differential equations of higher order:

$$y^{(n)} = f(t,y,y',y'',\ldots,y^{(n-1)}),$$

with $y(t_0) = a_0, y'(t_0) = a_1, \ldots, y^{(n-1)}(t_0) = a_{n-1}$, we proceed similarly using

$$||(y_1, y_2, \ldots, y_n) - (z_1, z_2, \ldots, z_n)|| = \left(\sum_{j=1}^{n} |y_j - z_j|^2 \right)^{\frac{1}{2}}.$$

Define functions

$$v_k = y^{(k)}, \quad k = 0, \ldots, n-1,$$

where $y^{(0)}$ is the function y. Then

$$v'_{n-1} = f(t, y, v_1, \ldots, v_{n-1}),$$

and for $j = 0, \ldots, n-2$

$$v'_j = v_{j+1}.$$

The operator T is given by

$$T(v_0, \ldots, v_{n-1}) = (w_0, \ldots, w_{n-1})$$

where

$$w_{j-1}(t) = a_{j-1} + \int_{t_0}^{t} v_j(s) ds$$

for $j = 1, \ldots, n-1$ and

$$w_{n-1}(t) = a_{n-1} + \int_{t_0}^{t} f(s, v_0(s), \ldots, v_{n-1}(s)) ds.$$

We leave it as an exercise to state the existence and uniqueness theorems for linear and non-linear n^{th} order (ordinary) differential equations. The proofs are the same as the proofs of the second order theorems, using the definitions above.