**Analytic solutions to linear differential equations**

In this note we prove that if $p$, $q$, and $r$ are analytic and have convergent power series expansions in $B_R = \{z : |z| < R\}$ then the solution $y$ to the second order linear differential equation

$$y'' = py' + qy + r \qquad y(0) = b_0, \quad y'(0) = b_1 \tag{1}$$

also is analytic and has a convergent power series expansion in $B_R$.

Suppose

$$p(z) = \sum_{n=1}^{\infty} p_n z^n, \quad q(z) = \sum_{n=1}^{\infty} q_n z^n, \quad r(z) = \sum_{n=1}^{\infty} r_n z^n$$

are power series expansions that converge in $B_R$. If $y(z) = \sum_{n=0}^{\infty} b_n z^n$ is an analytic solution of (1) then

$$n(n-1)b_n = \left(\sum_{j=1}^{n-1} j b_j p_{n-1-j}\right) + \left(\sum_{j=0}^{n-2} b_j q_{n-2-j}\right) + r_{n-2}. \tag{2}$$

It suffices to prove that if $b_n$ are defined by (2), then the series for $y$ converges in $B_R$ because series can be differentiated, multiplied and added on disks where they converge, without reducing the radius of convergence.

If $s < R$ then for some $M < \infty$,

$$\sum_{n=0}^{\infty} |p_k| s^k \le M \quad \text{and} \quad \sum_{n=0}^{\infty} |q_k| s^k \le M \quad \text{and} \quad \sum_{n=0}^{\infty} |r_k| s^k \le M. \tag{3}$$

Suppose $n_0 \ge (s + 2s^2)M + 1$. Choose $N$ with $1 \le N < \infty$ so that $|b_0| \le N$ and

$$j|b_j|s^j \le N \tag{4}$$

for $j = 1, \ldots, n_0$. We proceed by induction. Suppose $n > n_0$ and suppose (4) holds for $j < n$. Then by (2) and (4)

$$n(n-1)|b_n|s^n \le Ns\left(\sum_{j=1}^{n-1} |p_{n-1-j}| s^{n-1-j}\right) + Ns^2\left(\sum_{j=0}^{n-2} |q_{n-2-j}| s^{n-2-j}\right) + |r_{n-2}|s^n. \tag{5}$$

By (3) and (5)

$$n|b_n|s^n \le \frac{1}{n-1}(NsM + Ns^2M + s^2M) \le N\frac{sM + 2s^2M}{n-1} \le N.$$

By induction, (4) holds for all $j$, and hence the series for $y$ converges in $\{z : |z| < s\}$ by the root test. Since this is true for each $s < R$, the series for $y$ converges in $B_R$.

This argument has all of the ingredients to prove a similar result for $n^{th}$ order linear differential equations. Check your understanding by writing down a similar proof for first order equations.