

Consider the closed interval I . Let \mathcal{C} denote the set of all continuous functions on the interval I . For $Y \in \mathcal{C}$, set

$$\|Y\| = \max_{t \in I} |Y(t)|$$

For $Y_1, Y_2 \in \mathcal{C}$, we call $\|Y_1 - Y_2\|$ the *distance* between Y_1 and Y_2 . Notice that

$$(1) |Y_1(t) - Y_2(t)| \leq \|Y_2 - Y_1\|$$

$$(2) \|Y_1 - Y_2\| = 0 \text{ implies } Y_1 = Y_2.$$

(3) The triangle inequality holds:

$$\|Y_1 + Y_2\| \leq \|Y_1\| + \|Y_2\|.$$

A sequence $\{Y_n\}$ of functions in \mathcal{C} is said to *converge uniformly* to a function Y if for every $\epsilon > 0$, there is an integer N such that $\|Y_n - Y\| < \epsilon$ for all $n \geq N$.

A sequence $\{Y_n\}$ of continuous functions in \mathcal{C} is said to be a *Cauchy sequence (of continuous functions)* if for every $\epsilon > 0$, there is an integer N such that $\|Y_n - Y_m\| < \epsilon$ for all $n, m \geq N$.

Exercise. Convince yourself that if $\{Y_n\}$ is a Cauchy sequence in \mathcal{C} , then for all $t \in I$, the sequence $\{Y_n(t)\}$ is a Cauchy sequence of numbers, and therefore a convergent sequence.

Suppose that $\{Y_n\}$ is a Cauchy sequence of functions in \mathcal{C} . Define the limit of the sequence to be the function $Y : I \rightarrow \mathbb{R}$ defined by

$$Y(t) = \lim_{n \rightarrow \infty} Y_n(t). \quad (*)$$

(By the exercise above, this limit exists.) We will also write $Y = \lim_{n \rightarrow \infty} Y_n$.

Theorem 1. Let $\{Y_n\}$ a Cauchy sequence of functions, and let Y be the function defined in Equation (*). Then Y is continuous and the sequence $\{Y_n\}$ converges uniformly to Y .

Proof. Choose any $\epsilon > 0$. Because $\{Y_n\}$ is Cauchy, there exists an integer N such that $\|Y_n - Y_m\| < \epsilon/3$ for all $n, m \geq N$. It follows that

$$|Y_n(t) - Y_N(t)| \leq \|Y_n - Y_N\| < \epsilon/3$$

for all $n \geq N$ and all $t \in I$. Consequently

$$|Y(t) - Y_N(t)| = \left| \lim_{n \rightarrow \infty} Y_n(t) - Y_N(t) \right| = \lim_{n \rightarrow \infty} |Y_n(t) - Y_N(t)| \leq \epsilon/3,$$

for all $t \in I$. This shows that $\|Y - Y_N\| \leq \epsilon/3$. Now suppose $n > N$. Then by (3)

$$\|Y_n - Y\| \leq \|Y_n - Y_N\| + \|Y_N - Y\| \leq \epsilon/3 + \epsilon/3 < \epsilon.$$

This shows that $\{Y_n\}$ converges uniformly to Y .

We now show that Y is continuous. It suffices to show that for any $t \in I$ and any $\epsilon > 0$, there is a $\delta > 0$ such that $|Y(t) - Y(s)| < \epsilon$ for all $s \in I$ with $|s - t| < \delta$.

To see this, fix $t \in I$ and choose any $\epsilon > 0$. Then choose N so that $\|Y_n - Y\| < \epsilon/3$ for all $n \geq N$.

Since Y_N is continuous, there is a $\delta > 0$ such $|Y_N(s) - Y_N(t)| < \epsilon/3$ for all $s \in I$ with $|s - t| < \delta$.

Now suppose that $|s - t| < \delta$ and $s \in I$. Then

$$\begin{aligned} |Y(s) - Y(t)| &= |(Y(s) - Y_N(s)) + (Y_N(s) - Y_N(t)) + (Y_N(t) - Y(t))| \\ &\leq |Y(s) - Y_N(s)| + |Y_N(s) - Y_N(t)| + |Y_N(t) - Y(t)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus, Y is continuous. □

Next consider the subset $\mathcal{F} \subset \mathcal{C}$ of continuous functions defined as follows: For fixed numbers $t_0 \in I$, y_0 and $b > 0$,

$$\mathcal{F} = \{Y \in \mathcal{C} : Y(t_0) = y_0, \text{ and } |Y(t) - y_0| \leq b \text{ for all } t \in I\}.$$

Theorem 2. *Let $\{Y_n\}$ be a Cauchy sequence of functions in \mathcal{F} . Then the limit $Y = \lim_{n \rightarrow \infty} Y_n$ is also in \mathcal{F} .*

Proof. By the previous theorem, we know that Y is continuous, so we need only check that $Y(t_0) = y_0$ and $|Y(t) - y_0| \leq b$ for $t \in I$. But $Y(t_0) = \lim_{n \rightarrow \infty} Y_n(t_0) = y_0$; and since $|Y_n(t) - y_0| \leq b$ for all t and all n , clearly

$$|Y(t) - y_0| = \lim_{n \rightarrow \infty} |Y_n(t) - y_0| \leq b.$$

Thus $Y \in \mathcal{F}$. □

Definition. A map $T : \mathcal{F} \rightarrow \mathcal{F}$ (i.e. a rule that assigns to every function in \mathcal{F} another function in \mathcal{F}) is called a *contraction map* if there is a constant $0 < K < 1$ such that

$$\|T(Y_2) - T(Y_1)\| \leq K\|Y_2 - Y_1\|$$

for all $Y_1, Y_2 \in \mathcal{F}$.

Theorem 3. *Let $T : \mathcal{F} \rightarrow \mathcal{F}$ be a contraction map, let Y_0 be a function on \mathcal{F} , and let $\{Y_n\}$ be the sequence of functions in \mathcal{F} defined by $Y_{n+1} = T(Y_n)$. Then $\{Y_n\}$ is a Cauchy sequence. Moreover, if $Y = \lim_{n \rightarrow \infty} Y_n$, then $T(Y) = Y$, i.e. Y is a fixed point of T . Moreover, T has only one fixed point.*

Proof. Let $A = \|Y_1 - Y_0\|$. Since $Y_n = T^n(Y_0)$ and $Y_{n+1} = T^n(Y_1)$, we have the inequality

$$\|Y_{n+1} - Y_n\| \leq K^n \|Y_1 - Y_0\| = AK^n.$$

Now let $m = n + k > n$ and estimate as follows:

$$\begin{aligned} \|Y_{n+k} - Y_n\| &= \|Y_{n+k} - Y_{n+k-1} + Y_{n+k-1} - Y_{n+k-2} + \cdots + (Y_{n+1} - Y_n)\| \\ &\leq \|Y_{n+k} - Y_{n+k-1}\| + \|Y_{n+k-1} - Y_{n+k-2}\| + \cdots + \|Y_{n+1} - Y_n\| \\ &\leq AK^n(K^{k-1} + K^{k-2} + \cdots + K + 1) \leq \frac{AK^n}{1-K} \end{aligned}$$

To see that $\{Y_n\}$ is Cauchy, choose any $\epsilon > 0$ and choose N so that $\frac{AK^N}{1-K} < \epsilon$. Then by the above computation, for any $n, m \geq N$, $\|Y_n - Y_m\| < \epsilon$.

To see that $T(Y) = Y$, choose any $\epsilon > 0$. Set $Z = T(Y)$. We claim that $|Z(t) - Y(t)| < 2\epsilon$ for all $t \in I$. To see this, choose N so that $\|Y - Y_N\| < \epsilon$ and $\|Y - Y_{N+1}\| < \epsilon$. It follows from this that

$$\|Z - Y_{n+1}\| = \|T(Y) - T(Y_n)\| \leq K\|Y - Y_n\| < \epsilon.$$

But then, $|Z(t) - Y_{n+1}(t)| < \epsilon$ and $|Y(t) - Y_{n+1}(t)| < \epsilon$. Consequently,

$$|Z(t) - Y(t)| = |Z(t) - Y_{n+1}(t) + Y_{n+1}(t) - Y(t)| \leq |Z(t) - Y_{n+1}(t)| + |Y_{n+1}(t) - Y(t)| < \epsilon + \epsilon = 2\epsilon.$$

Since ϵ was arbitrary, it follows that $Z(t) = Y(t)$ for all t .

If W and Y are fixed points of T then

$$\|W - Y\| = \|T(W) - T(Y)\| \leq K\|W - Y\|.$$

Since $K < 1$, we must have $\|W - Y\| = 0$, and so $W = Y$. □