

FINAL EXAM SOLUTIONS, WINTER 2016

1. a. $\mathbf{r}'(t) = (-\sin t, \cos t, 2t)$

b. $\mathbf{r}''(t) = (-\cos t, -\sin t, 2)$

c. $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1+4t^2}$

d. $s(t) = \int_0^t \sqrt{1+4u^2} du$

e. $\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}$

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 2t \\ -\cos t & -\sin t & 2 \end{vmatrix} \\ &= \left(2\cos t + 2t\sin t, 2\sin t - 2t\cos t, \right. \\ &\quad \left. \underbrace{\sin^2 t + \cos^2 t}_1 \right)\end{aligned}$$

$$\begin{aligned}\|\mathbf{r}' \times \mathbf{r}''\|^2 &= 4\cos^2 t + 4t^2 \sin^2 t + 8t \cos t \sin t \\ &\quad + 4\sin^2 t + 4t^2 \cos^2 t - 8t \cos t \sin t + 1 \\ &= 5 + 4t^2\end{aligned}$$

so $\kappa = \frac{(5+4t^2)^{1/2}}{(1+4t^2)^{3/2}}$

$$f. \quad T = \frac{N'}{\|N'\|} = \left(\frac{-\sin t}{\sqrt{1+4t^2}}, \frac{\cos t}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right)$$

$$g. \quad B = \frac{N' \times N''}{\|N' \times N''\|} = \left(\frac{2\cos t + 2t \sin t}{\sqrt{5+4t^2}}, \frac{2\sin t - 2t \cos t}{\sqrt{5+4t^2}}, \frac{1}{\sqrt{5+4t^2}} \right)$$

$$h. \quad N = B \times T = \begin{pmatrix} \frac{-\cos t + 4t \sin t - 4t^2 \cos t}{\sqrt{1+4t^2} \sqrt{5+4t^2}} & \frac{-\sin t - 4t \cos t - 4t^2 \sin t}{\sqrt{1+4t^2} \sqrt{5+4t^2}} \\ \frac{2\sin^2 t - 2t \sin t \cos t + 2\cos^2 t + 2t \sin t \cos t}{\sqrt{1+4t^2} \sqrt{5+4t^2}} & \end{pmatrix}$$

$$i. \quad B \cdot (0, 0, 1) = 1 \cdot 1 \cdot \cos \alpha = \frac{1}{\sqrt{5+4t^2}}$$

$$\alpha = \cos^{-1} \left(\frac{1}{\sqrt{5+4t^2}} \right) \quad (\text{where } 0 < \alpha < \pi/2)$$

$$2. (s^2 + 4s + 13) \mathcal{L}(y) = \mathcal{L}(e^{-t^2})$$

$$\mathcal{L}(y) = \frac{1}{(s^2 + 4s + 4) + 9} \mathcal{L}(e^{-t^2})$$

$$= \frac{1}{(s+2)^2 + 9} \mathcal{L}(e^{-t^2})$$

By the table $\mathcal{L}(e^{-2t} \sin 3t) = \frac{3}{(s+2)^2 + 9}$

$$\therefore \mathcal{L}(y) = \mathcal{L}\left(\frac{1}{3} e^{-2t} \sin 3t\right) \mathcal{L}(e^{-t^2})$$

By the table again:

$$y(t) = \int_0^t \frac{1}{3} e^{-2x} \sin 3x \cdot e^{-(t-x)^2} dx$$

$$= \frac{1}{3} \int_0^t e^{-x^2} \sin 3x dx$$

other forms {

$$= \int_0^t \frac{1}{3} e^{-2(t-x)} \sin 3(t-x) e^{-x^2} dx$$

$$3. (1+x+x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$= \sum \left[(n+2)(n+1)a_{n+2} + (n+1)n a_{n+1} + n(n-1)a_n - 2na_n + 12a_n \right] x^n$$

$$\text{so } (n+2)(n+1)a_{n+2} + (n+1)n a_{n+1} + (n^2 - 3n + 12)a_n = 0$$

④.

$$a_{n+2} = \frac{-(n+1)n a_{n+1} - (n^2 - 3n + 12)a_n}{(n+1)(n+2)}$$

$$a_0 = y(0) = 1 \quad a_1 = y'(0) = 2$$

$$n=0 \quad a_2 = \frac{-12a_0}{2} = -6$$

$$n=1 \quad a_3 = \frac{-2a_2 - 10a_1}{6} = \frac{12 - 20}{6} = -\frac{4}{3}$$

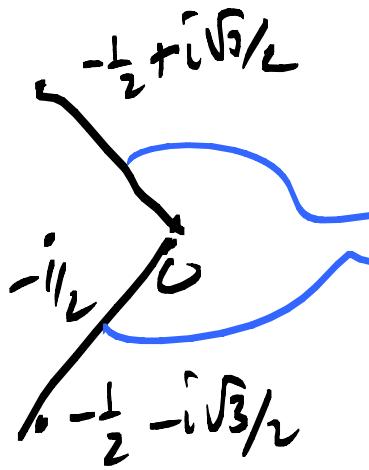
$$n=2 \quad a_4 = \frac{-6a_3 - 10a_2}{12} = \frac{8 + 60}{12} = \frac{17}{3}$$

$$④ T_4(x) = 1 + 2x - 6x^2 - \frac{4}{3}x^3 + \frac{17}{3}x^4$$

$$\textcircled{C} \quad y'' - \frac{2x}{1+x+x^2}y' + \frac{12}{1+x+x^2}y = 0$$

The coefficients have a power series expansion convergent in the largest disk on which the denominators are $\neq 0$, since these are rational functions.

$$1+x+x^2 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$



$$\text{distance} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

so the series solution converges on $\{z \in \mathbb{C} : |z| < 1\}$ which contains the interval $(-1, 1)$

d. By the existence and uniqueness theorem again,
 a series solution expanded about $x_0 = 5$
 converges on $|z - 5| < \text{dist}(5, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)$

$$= \sqrt{\left(5 + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{26 + 5} = \sqrt{31}$$

for any initial conditions.

e. $W(y_1, y_2)(5) = y_1'(5)y_2'(5) - y_1'(5)y_2(5) = 1$

By a theorem proved in the book, $W(y_1, y_2)(x) \neq 0$

for $5 - \sqrt{31} < x < 5 + \sqrt{31}$ (where the solution is valid)

This includes 0, so $W(y_1, y_2)(0) \neq 0$

Now solve $1 = c_1 y_1(0) + c_2 y_2(0) \quad ①$

$$2 = c_1 y_1'(0) + c_2 y_2'(0) \quad ②$$

Mult. 1st eq by $y_2'(0)$

Mult 2nd eq by $y_2(0)$

Subtract

$$y_2'(0) = c_1 y_1(0) y_2'(0) + c_2 y_2(0) y_2'(0)$$

$$2y_2(0) = c_1 y_1'(0) y_2(0) + c_2 y_2'(0) y_2(0)$$

$$y_2'(0) - 2y_2(0) = c_1 (y_1(0) y_2'(0) - y_1'(0) y_2(0))$$

$$\text{Because } W(y_1, y_2)(0) \neq 0 : \quad c_1 = \frac{y_2'(0) - 2y_2(0)}{W(y_1, y_2)(0)}$$

Similarly mult ① $\times y_1'(0)$

mult ② $\times y_1(0)$ & subtract

$$y_1'(0) - 2y_1(0) = c_2 [y_2(0) y_1'(0) - y_1(0) y_2'(0)]$$

$$\text{Again because } W(y_1, y_2)(0) \neq 0 : \quad c_2 = \frac{2y_1(0) - y_1'(0)}{W(y_1, y_2)(0)}$$

Thus y and $c_1 y_1 + c_2 y_2$ are solutions on an interval containing 0 with the same value & derivative at 0

By uniqueness $y = c_1 y_1 + c_2 y_2$ on the largest interval containing 0 on which both sides converge.