11.6: 5, 9, 13, 21, 37

11.7: 7, 13, 21, 38

12.1: 13, 14, 21, 22, 23, 24;

12.2: 1, 7, 13, 15, 17, 18, 29, 32, 36;

12.3: 1, 3, 11, 15, 27, 29, 35;

12.4: 1, 3, 4, 15, 32, 41, 48

12.5: 2, 5, 15, 27, 29, 33, 35.

To hand in:

(1) Let $\{a_n\}$ be the sequence defined inductively by $a_1 = 1$, $a_{n+1} = \dfrac{1}{a_n^4 + 16}$.

(a) Show that $\{a_n\}$ is a Cauchy sequence.

(b) Show that $\{a_n\}$ converges to a solution of the equation $x^5 + 16x - 1 = 0$.

(c) Show that if $\{b_n\}$ is the sequence defined by $b_1 = 2$, $b_{n+1} = \dfrac{1}{b_n^4 + 16}$, then $\{b_n\}$ is convergent and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n$.

Hint: *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/(x^4 + 16)$.*

(2) Show that the equation

$$x = 1 + \int_0^x \frac{\cos(t)}{t^2 + 4}\, dt$$

has one and only one solution.

(3) Recall that (by definition) $\lim\limits_{x\to\infty} f(x) = L$ if and only if for every real number $\epsilon > 0$ there is a real number x_0 such that $|f(x) - L| < \epsilon$ for all $x > x_0$.

Prove the following:

$$\lim_{x\to\infty} f(x) = L \text{ if and only if } \lim_{t\to 0^+} f(1/t) = L\,.$$

(4) Show that for any real number c,

$$\lim_{x\to\infty} \left(\frac{x+c}{x-c}\right)^x = e^{2c}\,.$$

(5) Use the hint given in class.

Show that $\displaystyle\int_2^\infty \frac{dx}{x(\ln(x))^p}$ converges if and only if $p > 1$.

(b) Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges and that its sum equals 2.1 to one decimal place (i.e. with an error less than 0.05).

(c) Show that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)}$ diverges, but that

$$8.10 \leq \sum_{k=2}^{10^{1000}} \frac{1}{k(\ln k)} \leq 8.83$$

as you can see, this series diverges *very* slowly! If you tried to detect the divergence experimentally by adding a few million terms on the computer, you would not succeed.

(6) Determine whether the series $\displaystyle\sum_{k=1}^{\infty} k! \left(\frac{5}{2k}\right)^k$ converges or diverges.